CLASSICAL ORTHOGONAL POLYNOMIALS AND THEIR ASSOCIATED FUNCTIONS IN COMPLEX DOMAIN

Peter Rusev

CONTENTS

PREFACE

The importance of the classical orthogonal polynomials of Jacobi, Laguerre and Hermite for the contemporary mathematics, as well as for wide range of their applications in the physics and engineering, is beyond any doubt.

It is well-known that these polynomials play an essential role in problems of the approximation theory. They occur in the theory of differential and integral equation as well as in the mathematical statistics. Their applications in the quantum mechanics, scattering theory, automatic control, signal analysis and axially symmetric potential theory are also known.

At the focus of the present book are the classical orthogonal polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$, $\{L_n^{(\alpha)}(z)\}_{n=o}^{\infty}$ and $\{H_n(z)\}_{n=0}^{\infty}$ of Jacobi, Laguerre and Hermite and their associated functions $\{Q_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$, $\{M_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ and $\{G_n(z)\}_{n=0}^{\infty}$, mainly as tools for representations of holomorphic functions. The core of the discussions are the problems arising in expanding of holomorphic functions in series of the *Jacobi, Laguerre and Hermite systems*, i.e., in series of Jacobi, Laguerre and Hermite polynomials and in their associated functions, as well as the applications of the obtained results in the complex analysis.

Following Tricomi, in Chapter I we introduce the classical orthogonal polynomials by means of Rodrigues' type formulas. This is just one possible way to define these polynomials without any restriction on the parameters α, β in the case of Jacobi, and on α in the case of Laguerre polynomials.

Since the weight functions for the Jacobi, Laguerre and Hermite polynomials satisfy differential equations of Pearson's type, it follows immediately that these polynomials are solutions of differential equations of hypergeometric type. This means that the classical orthogonal polynomials appear also as particular cases of the hypergeometric functions.

The orthogonality of Jacobi, Laguerre and Hermite polynomials with respect to corresponding weight functions on suitable rectifiable Jordan curves in the (extended) complex plane is established under the single assumption that the complex numbers $\alpha+1, \beta+1$ and $\alpha+\beta+1$ are not equal to $0, -1, -2, \ldots$. As a corollary, it is shown that each of the systems $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$, $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ and $\{H_n(z)\}_{n=0}^{\infty}$ is a solution of a linear recurrence equation of second order.

The associated Jacobi, Laguerre and Hermite functions are defined, by means of suitable Cauchy type integral transforms, as "second" solutions of the corresponding recurrence equations. Then, as a corollary, the existence of Christoffel-Darboux type formulas for the Jacobi, Laguerre and Hermite systems is established.

Chapter II presents various and mostly familiar integral representations of the classical orthogonal polynomials and associated functions. Among them, e.g., are those for the Laguerre polynomials and associated functions, which involve Bessel functions of different kind. From the numerous generating functions we have chosen only those used in the next Chapters. Some of them, and especially those for the Jacobi polynomials, have more general form than those occasionally occur in the literature.

The asymptotics of the classical orthogonal polynomials and associated functions is studied in Chapter III. The classical Darboux' method is applied to the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ and, as a result, we have an asymptotic formula for these polynomials in the region $\mathbb{C} \setminus [-1,1]$, when the parameters α, β are arbitrary complex numbers. The asymptotic formula for the Jacobi associated functions $\{Q_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ in the same region is obtained using their expansions in terms of the powers of the corresponding inverse of Zhukovskii function provided $\alpha+1, \beta+1$ and $\alpha+\beta+2$ are not equal to $0, -1, -2, \dots$.

Using Szegö's asymptotic formula for the Hermite polynomials as well as Uspensky's integral representation of Laguerre polynomials in terms of the even Hermite polynomials, we derive an asymptotic formula for the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ in the region $\mathbb{C} \setminus [0,\infty)$, with no restriction on the parameter α.

In Chapter IV we use the asymptotic formulas and inequalities for the systems of classical orthogonal polynomials and their associated functions in order to give a full description of the regions and the mode of convergence of series in these systems. The existence of Cauchy-Hadamard's formulas as well as Abel's type theorems confirms that there is a close analogy with the power series. For example, it is proved that the regions of absolute convergence of the series under consideration coincide with their regions of convergence.

In Chapter IV we consider also the problem of uniqueness of the representations in Jacobi, Laguerre and Hermite series. In general, the orthogonal expansions have the uniqueness property and we elucidate the difficulties arising when this property has to be proved for series in Jacobi, Laguerre and Hermite polynomials. The uniqueness of the series expansions in the corresponding systems of associated functions follows from the asymptotic formulas for these systems only.

Chapter V is the central one in the book. The main problem we study there is to describe the $\mathbb{C}$-vector spaces of holomorphic functions possessing series representations in the Jacobi, Laguerre and Hermite systems.

Denote by $E(r),\ 1 < r < \infty$, the interior of the ellipse $e(r)$ with focuses at the points -1 and 1 and with the sum of its semiaxes equal to r. Let $E^*(r), 1 < r < \infty$, be the exterior of $e(r)$ with respect to the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, i.e., $E^*(r) = \overline{\mathbb{C}} \setminus \overline{E(r)}$, and assume that $E(\infty) = \mathbb{C}$ and $E^*(1) = \overline{\mathbb{C}} \setminus [-1,1]$.

It is a familiar fact that if $1 < r \le \infty$, then the system of Jacobi polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ is a basis of the space $\mathcal{H}(E(r))$ of the complex-valued functions holomorphic in $E(r)$. Likewise, if $1 \le r < \infty$, then the system of Jacobi associated functions $\{Q_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ is a basis of the space of the complex-valued functions holomorphic in the region $E^*(r)$ and vanishing at the point of infinity. Let us note that these properties of the Jacobi systems are established in Chapter V when the parameters α, β are arbitrary complex numbers such that $\alpha+1, \beta+1, \alpha+\beta+2$ are not equal to $0, -1, -2, \ldots$.

Let $0 < \tau_0 < \infty$ and $S(\tau_0)$ be the strip defined by the inequality $|\operatorname{Im} z| < \tau_0$, and assume that $S(\infty) = \mathbb{C}$. Using the inequalities for the Hermite polynomials in the complex plane, included in Chapter III, it is easy to prove that the space $\mathcal{H}(\tau_0)$ of complex-valued functions holomorphic in $S(\tau_0)$ and having there representations by (convergent) series of Hermite polynomials is a proper subspace of the space $\mathcal{H}(S(\tau_0))$ of complex-valued functions holomorphic in the strip $S(\tau_0)$.

E. Hille gave in 1940 a growth characterization of the functions in the space $\mathcal{H}(\tau_0)$. He proved that a function $f \in \mathcal{H}(S(\tau_0))$ belongs to the space $\mathcal{H}(\tau_0)$ if and only if for each $\tau \in [0, \tau_0)$ there exists a positive constant $K = K(\tau)$ such that $|f(z)| = |f(x+iy)| \le K \exp\{x^2/2 - |x|(\tau^2 - y^2)^{1/2}\}$ whenever $|\operatorname{Im} z| \le \tau$. A complete proof of Hille's theorem is included in Chapter V.

Let $\Delta(\lambda_0), 0 < \lambda_0 < \infty$, be the interior of the parabola $p(\lambda_0)$ with focus at the origin and vertex at the point $-\lambda_0^2$, and assume that $\Delta(\infty) = \mathbb{C}$. Denote by $\mathcal{L}^{(\alpha)}(\lambda_0)$ the space of complex-valued functions holomorphic in the region $\Delta(\lambda_0)$ and expandable in (convergent) series of the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ there.

Using partly Hille's theorem, H. Pollard solved in 1947 the problem of finding a growth description of the functions in the space $\mathcal{L}^{(\alpha)}(\lambda_0)$ when $\alpha = 0$. In fact, he proved that a function $f \in \mathcal{H}(\Delta(\lambda_0))$ belongs to the space $\mathcal{L}^{(0)}(\lambda_0)$ if and only if for each $\lambda \in [0, \lambda_0)$ there exists a positive constant $L = L(\lambda)$ such that if $\operatorname{Re}(-z)^{1/2} \le \lambda$, then $f(z)| = |f(x+iy)| \le L \exp(\varphi(\lambda; x, y))$, where

$$\varphi(\lambda; x, y) = \frac{\sqrt{x^2+y^2}+x}{4} - \left[\frac{\sqrt{x^2+y^2}+x}{2}\left(\lambda^2 - \frac{\sqrt{x^2+y^2}-x}{2}\right)\right]^{1/2}$$

The validity of Pollard's theorem for every $\alpha > -1$ was established by O. Szász and N. Yeardley in 1958. Our approach to the same problem is based on the Uspenski integral transform, which in fact is a fractional operator of Riemann-Liouville's type. We prove that Pollard's theorem is valid under the only assumption that the complex parameter α is not of the kind $k+1/2+i\eta$ with $k \in \mathbb{Z}$ and $\eta \in \mathbb{R}^* = \mathbb{R}\setminus\{0\}$, as well as that $\alpha+1 \in \mathbb{C}\setminus\mathbb{Z}^-$. In particular, it holds for each real α different from $-1, -2, -3, \ldots$. The case $\alpha = k+1/2+i\eta$ with $k \in \mathbb{Z}$ and $\eta \in \mathbb{R}^*$ is still an open problem.

Let us note that the growth description of Hille's or Pollard's type of the holomorphic functions having representations by series in Hermite or Laguerre associated functions, is also still open.

The last section of Chapter V is devoted to the problem of holomorphic extension of complex-valued functions defined on intervals of the real line or on smooth Jordan curves in the extended complex plane passing trough the point at infinity.

The main topic in Chapter VI is the series representations of holomorphic functions by means of Laguerre and Hermite systems in terms of classical integral transform. First we discuss how the integral representation of Laguerre polynomials by means of Bessel functions of the first kind allows to obtain the spaces $\mathcal{L}^{(\alpha)}(\lambda_0), 0 < \lambda_0 \leq \infty$ as Hankel's type integral transforms of suitable spaces of entire functions of exponential type.

Let $\Delta^*(\mu_0) := \mathbb{C} \setminus \overline{\Delta(\mu_0)}$ if $0 < \mu_0 < \infty$ and $\Delta^*(0) := \mathbb{C} \setminus [0, \infty)$. Denote by $\mathcal{M}^{(\alpha)}(\mu_0)$ the space of complex functions holomorphic in the region $\Delta^*(\mu_0)$ and having representations by (convergent) series in the Laguerre associated functions $\{M_n^{(\alpha)}(z)\}_{n=0}^{\infty}$. Then, in Chapter VI, we prove that each of the spaces $\mathcal{M}^{(\alpha)}(\mu_0), 0 \leq \mu_0 < \infty$, is the image of a space of entire functions of exponential type under a Meijer's type integral transform.

On the basis of the relations between Laguerre and Hermite systems, we obtain further corresponding results for the Hermite series. More precisely, we emphasize that the holomorphic functions, representable by series in Hermite polynomials or associated functions, are Fourier transforms of suitable entire functions.

In the final part of Chapter VI we discuss the expansion in series of Laguerre and Hermite polynomials of entire functions of exponential type in terms of their indicator functions.

In the first two Sections of Chapter VII the main attention is paid to the convergence and Cesaro's summability of positive order of series in the Jacobi and Laguerre polynomials on the boundaries of their regions of convergence. The results obtained are analogous to well-known theorems for the Fourier series. A typical one is a (C, δ)-version of the classical Feijer's theorem for a class of series in the Laguerre polynomials.

In the last Section of Chapter VII it is shown that a classical theorem of Fatou for the power series can be extended to series in the Laguerre and Hermite polynomials and associated functions. Especially, in the case of series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$, it is to be noted the unexpected fact that the corresponding Fatou type condition on the growth of their coefficients involves the parameter α.

Without any pretentions for completeness, a review on singular points and analytical continuation of holomorphic functions defined by series in the classical orthogonal polynomials is included as a final part of the book. It is simply an effort to make a bridge between classical theorems about power series and series in Jacobi, Laguerre and Hermite polynomials.

The theory of the classical orthogonal polynomials and their associated functions in the complex domain is a "great user" of special functions. For readers, whose familiarity with them is restricted, in the Appendix all we need in this book from the theory of classical special functions is given.

Chapter I

JACOBI, LAGUERRE AND HERMITE POLYNOMIALS AND ASSOCIATED FUNCTIONS

1. Pearson's differential equation

1.1 Let A and $B \not\equiv 0$ be polynomials with complex coefficients such that $\deg A \leq 1$ and $\deg B \leq 2$. If $\deg B = 2$, then B has two roots z_1 and z_2 in the complex plane which may be distinct or coincident. If $\deg B = 1$, then we still assume that the polynomial B has two roots: one at $z_1 \in \mathbb{C}$ and the other one at the point of infinity. If $B \equiv \text{Const} \neq 0$, then we agree that both z_1 and z_2 coincide with the point ∞. If $A \not\equiv 0$, then we assume the same for the polynomial A.

(I.1.1) *The differential equation*

$$B(z)w' - A(z)w = 0 \tag{1.1}$$

has holomorphic nonzero solutions in each simply connected region $G \subset \mathbb{C}\backslash\{z_1, z_2\}$, i.e. in each simply connected subregion of the complex plane not containing finite roots of the polynomial B.

Proof. Since the region G is simply connected and the function $A(z)/B(z)$ is holomorphic there, it has a primitive in G. Such is the function

$$f(z) = \int_{z_0}^{z} (A(\zeta)/B(\zeta))\, d\zeta,$$

where z_0 is a fixed point of $G, z \in G$ and the path of integration is an arbitrary rectifiable curve lying entirely in G and connecting the points z_0 and z.

A direct computation yields that the function w defined by $w(z) = \exp\{f(z)\}$ for $z \in G$ is a solution of the equation (1.1) in G and, moreover, $w(z) \neq 0$ whenever $z \in G$.

Remark. Each holomorphic solution w of the equation (1.1) in a region not containing the roots of the polynomial B is either nowhere zero or it is identically zero. Indeed, if $w(z_0) = 0$ at a point z_0 of such a region, then successive differentiation followed by replacing z by z_0 leads to $w^{(k)}(z_0) = 0,\ k = 1, 2, 3, \ldots$ and, hence, $w \equiv 0$.

(I.1.2) *Suppose that $w(z) \not\equiv 0$ is a solution of the equation* (1.1) *in the region $G \subset \mathbb{C} \setminus \{z_1, z_2\}$. Then for each $n = 0, 1, 2, \ldots$ the function W_n defined by*

$$W_n(z) = \frac{1}{w(z)} \frac{d^n}{dz^n}\{B^n(z)w(z)\} \tag{1.2}$$

is a polynomial of degree at most n.

Proof. Define $A_0(z) \equiv 1$ and $A_1(z) = A(z)$. Then

$$\frac{w^{(k)}(z)}{w(z)} = \frac{A_k(z)}{B^k(z)},\ z \in G,\ k = 0, 1, 2, \dots, \tag{1.3}$$

where A_k is a polynomial of $\deg A_k \le k$. Indeed, (1.3) is satisfied when $k = 0, 1$. Suppose that (1.3) holds for some $k \ge 2$. By differentiation of (1.3) we obtain

$$\frac{w^{(k+1)}(z)}{w(z)} - \frac{w^{(k)}(z)}{w(z)} \cdot \frac{w'(z)}{w(z)} = \frac{A_k'(z)B(z) - kA_k(z)B'(z)}{B^{k+1}(z)}.$$

Then, having in mind (1.1) and (1.3), we can write

$$\frac{w^{(k+1)}(z)}{w(z)} = \frac{A_k'(z)B(z) - kA_k(z)B'(z) + A(z)A_k(z)}{B^{k+1}(z)}.$$

Let us denote $A_{k+1}(z) = A_k'(z)B(z) - kA_k(z)B'(z) + A(z)A_k(z)$. Since, by assumption $\deg A_k \le k$, we conclude that $\deg A_{k+1} \le k + 1$.

Likewise we prove that for each $k = 0, 1, 2, \dots, n$,

$$(B^n(z))^{(n-k)} = B^k(z)B_{n,k}(z), \tag{1.4}$$

where $B_{n,k}$ is a polynomial of $\deg B_{n,k} \le n - k$.

Since

$$W_n(z) = \sum_{k=0}^{n} \binom{n}{k} A_k(z)B_{n,k}(z), \tag{1.5}$$

the desired assertion follows from (1.3), (1.4) and the Leibniz rule.

Remark. The polynomials $\{W_n(z)\}_{n=0}^{\infty}$, defined by the equalities (1.2) in the region G, depend neither on G nor on the solution $w(z)$ of the equation (1.1). They are uniquely determined by the coefficients of the polynomials A and B.

(I.1.3) *In the cases*:

$$\begin{aligned}
&C_1) \quad \infty \ne z_1 \ne z_2 \ne \infty;\\
&C_2) \quad z_1 \ne \infty,\ z_2 = \infty, \deg A = 1;\\
&C_3) \quad z_1 = z_2 = \infty, \deg A = 1;\\
&C_4) \quad z_1 = z_2 \ne \infty, A(z_1) \ne 0,
\end{aligned}$$

the equation (1.1) *can be transformed by means of a linear substitution of the independent variable* ($z \longmapsto \lambda z + \mu$) *to exactly one of the following forms*:

$$(C_1)\ \frac{w'}{w'} = -\frac{\alpha}{1-z} + \frac{\beta}{1+z};$$
$$(C_2)\ \frac{w'}{w} = \frac{\alpha - z}{z};$$
$$(C_3)\ \frac{w'}{w} = -2z;$$
$$(C_4)\ \frac{w'}{w} = \frac{mz-1}{z^2}.$$

We leave the proof of the above assertion to the reader as a simple exercise.

1.2 The classical polynomials of Jacobi, Laguerre and Hermite are related to the cases C_1), C_2) and C_3), respectively. For particular choice of the region G in each of these cases, the corresponding solutions of the equations (C_1), (C_2) and (C_3) have the form:

$$C_1)\ G = G_1 = \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}, \quad w(z) = (1-z)^\alpha (1+z)^\beta;$$
$$C_2)\ G = G_2 = \mathbb{C} \setminus (-\infty, 0], \quad w(z) = z^\alpha \exp(-z);$$
$$C_3)\ G = G_3 = \mathbb{C}, \quad w(z) = \exp(-z^2),$$

where, by definition, $(1-z)^\alpha(1+z)^\beta = \exp\{\alpha \log(1-z) + \beta \log(1+z)\}$ and $z^\alpha = \exp(\alpha \log z)$.

(I.1.4) **(a)** *If* $\alpha+\beta+2 \notin \mathbb{Z}^- \cup \{0\}$, i.e. $\alpha+\beta+2$ *is not equal to* $0, -1, -2, \dots,$ *then for each* $n = 0, 1, 2, \dots$ *the polynomial* (1.2) *with* $B(z) = 1 - z^2$ *and* $w(z) = (1-z)^\alpha(1+z)^\beta$, $z \in G_1$, *is of degree* n;

(b) *for each* $\alpha \in \mathbb{C}$ *and* $n = 0, 1, 2, \dots$ *the polynomial* (1.2) *with* $B(z) = z$ *and* $w(z) = z^\alpha \exp(-z)$, $z \in G_2$, *is of degree* n;

(c) *for each* $n = 0, 1, 2, \dots$ *the polynomial* (1.2) *with* $B(z) \equiv 1$ *and* $w(z) = \exp(-z^2)$ *is of degree* n.

Proof. **(a)** In this case

$$W_n(z) = \sum_{k=0}^{n} A_{n,k}(1-z)^k(1+z)^{n-k}, \ n = 0, 1, 2, \dots,$$

where

$$A_{n,k} = (-1)^{n-k}\binom{n}{k}(n-k)!\binom{n+\alpha}{n-k}k!\binom{n+\beta}{k}, \ k = 0, 1, 2, \dots, n.$$

Therefore,

$$(1.6) \qquad (-1)^n (n!)^{-1} W_n(z)$$

$$= \sum_{k=0}^{\infty} (-1)^k \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (1-z)^k (1+z)^{n-k}, \ n = 0, 1, 2, \ldots.$$

From the last representation it follows that if $n \geq 1$, then the coefficient of z^n in $W_n(z)$ is equal to

$$(1.7) \qquad (-1)^n n! \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k}.$$

Further we use the combinatorial identity

$$(1.8) \qquad \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} = \binom{2n+\alpha+\beta}{n}.$$

It can be obtained by multiplication of the two power series

$$(1+z)^{n+\alpha} = \sum_{p=0}^{\infty} \binom{n+\alpha}{p} z^p, \quad |z| < 1,$$

$$(1+z)^{n+\beta} = \sum_{q=0}^{\infty} \binom{n+\beta}{q}, \quad |z| < 1.$$

We have

$$(1+z)^{2n+\alpha+\beta} = \sum_{s=0}^{\infty} \Bigg\{ \sum_{p+q=s} \binom{n+\alpha}{p} \binom{n+\beta}{q} \Bigg\} z^s, \quad |z| < 1.$$

But on the other hand

$$(1+z)^{2n+\alpha+\beta} = \sum_{s=0}^{\infty} \binom{2n+\alpha+\beta}{s} z^s, \quad |z| < 1$$

and, hence,

$$\sum_{p+q=n} \binom{n+\alpha}{p} \binom{n+\beta}{q} = \binom{2n+\alpha+\beta}{n}.$$

Then from (1.7) and (1.8) it follows that the coefficient of z^n in $W_n(z)$ is equal to

$$(1.9) \qquad (-1)^n n! \binom{2n+\alpha+\beta}{n}, \quad n = 1, 2, 3, \ldots.$$

Since $\alpha+\beta+2 \notin \mathbb{Z}^- \cup \{0\}$, this coefficient is different from zero for each $n = 1,2,3,\dots$. But from $W_0(z) \equiv 1$ it is seen that the same holds for $n=0$.

(b) The Leibniz rule gives in this case that

$$W_n(z) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+\alpha}{n-k} z^k, \ n = 0,1,3,\dots. \tag{1.10}$$

From the above representation it follows that the coefficient of z^n is

$$(-1)^n, \quad n = 0,1,2,\dots, \tag{1.11}$$

i.e., it is always different from zero.

(c) Let $\tilde{\gamma}$ be a positively oriented circle centered at the point z. Then

$$W_n(z) = \frac{n!}{2\pi i \exp(-z^2)} \int_{\tilde{\gamma}} \frac{\exp(-\zeta^2)}{(\zeta - z)^{n+1}} \, d\zeta, \ n = 0,1,2,\dots.$$

After the "translation" $\zeta \longmapsto \zeta + z$ we obtain

$$W_n(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{\exp(-\zeta^2 - 2z\zeta)}{\zeta^{n+1}} \, d\zeta, \ n = 0,1,2,\dots, \tag{1.12}$$

where γ is a positively oriented circle with center at the origin.

Since

$$\exp(-\zeta^2 - 2z\zeta) = \exp(-\zeta^2)\exp(-2z\zeta)$$

$$= \sum_{k=0}^{\infty} \frac{(-1^k)\zeta^{2k}}{k!} \sum_{k=0}^{\infty} \frac{(-1)^k (2z\zeta)^k}{k!} = \sum_{\nu=0}^{\infty} \left(\sum_{k=0}^{[\nu/2]} \frac{(-1)^{\nu-k}(2z)^{\nu-2k}}{k!(\nu-2k)!} \right) \zeta^\nu$$

for each $\zeta \in \mathbb{C}$, (1.9) yields that

$$W_n(z) = (-1)^n n! \sum_{k=0}^{[n/2]} \frac{(-1)^k (2z)^{n-2k}}{k!(n-k)!}, \ n = 0,1,2,\dots. \tag{1.13}$$

Obviously, in this case the coefficient of z^n in W_n is equal to

$$(-1)^n 2^n, \ n = 0,1,2,\dots, \tag{1.14}$$

and, hence, $\deg W_n = n, \ n = 0,1,2,\dots$.

2. Definition of Jacobi, Laguerre and Hermite polynomials

2.1 In the case C_1) the polynomilas $(-1)^n 2^{-n}(n!)^{-1}W_n(z), n = 0,1,2,\dots$ with $\alpha+\beta+2 \notin \mathbb{Z}^- \cup \{0\}$ are called Jacobi polynomials with parameters α and β. Usually, they are denoted by $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$. From this definition it follows that

$$(2.1)\qquad P_n^{(\alpha,\beta)}(z) = \frac{(-1)^n}{n!2^n(1-z)^\alpha(1+z)^\beta}\frac{d^n}{dz^n}\{(1-z)^{n+\alpha}(1+z)^{n+\beta}\},$$

$$z \in \mathbb{C}\setminus\{(-\infty,0]\cup[0,\infty)\},\ n = 0,1.2,\dots.$$

Then (1.6) enables to get the representation

$$(2.2)\qquad P_n^{(\alpha,\beta)}(z) = \sum_{n=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\binom{n+\beta}{k}\Big(\frac{1-z}{2}\Big)^k\Big(\frac{1+z}{2}\Big)^{n-k},$$

$$z\in\mathbb{C},\ n = 0,1,2,\dots.$$

Denote by $p_n^{(\alpha,\beta)},\ n = 0,1,2,\dots$ and $q_n^{(\alpha,\beta)},\ n = 1,2,3,\dots,$ the coefficients of z^n and z^{n-1}, respectively, in the n-th Jacobi polynomial with parameters α and β. Then (1.10) with $\lambda_n = (-1)^n2^{-n}(n!)^{-1}$ yields

$$(2.3)\qquad p_n^{(\alpha,\beta)} = 2^{-n}\binom{2n+\alpha+\beta}{n},\ n = 0,1,2,\dots.$$

In order to calculate $q_n^{(\alpha,\beta)}$, we use the polynomial

$$\tilde{P}_n^{(\alpha,\beta)}(z) = z^nP_n^{(\alpha,\beta)}(1/z) = \sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\binom{n+\beta}{k}\Big(\frac{z-1}{2}\Big)^k\Big(\frac{z+1}{2}\Big)^{n-k}.$$

Obviously,

$$(2.4)\qquad q_n^{(\alpha,\beta)} = \Big\{\frac{d}{dz}\tilde{P}_n^{(\alpha,\beta)}(z)\Big\}_{z=0},\ n = 1,2,3,\dots,$$

and as a result of (1.9) and (2.4) we obtain

$$(2.5)\ q_n^{(\alpha,\beta)} = n2^{-n}\binom{2n+\alpha+\beta}{n} - 2^{-n+1}\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k},\ n = 1,2,3,\dots.$$

From the expansions

$$z(n+\beta)(1+z)^{n+\beta-1} = \sum_{q=0}^{\infty}q\binom{n+\beta}{q},\ |z|<1,$$

$$z(n+\beta)(1+z)^{\alpha}(1+z)^{n+\beta-1} = \sum_{s=0}^{\infty}\Big\{\sum_{p+q=s} q\binom{n+\alpha}{p}\binom{n+\beta}{q}\Big\}z^s,\ |z|<1,$$

$$z(n+\beta)(1+z)^{2n+\alpha+\beta-1} = \sum_{s=0}^{\infty}(n+\beta)\binom{2n+\alpha+\beta-1}{s}z^{s+1},\ |z|<1,$$

it follows

$$\sum_{k=0}^{n} k\binom{n+\alpha}{n-k}(n+\beta k) = (n+\beta)\binom{2n+\alpha+\beta-1}{n-1},\ n=1,2,3,\dots.$$

After some standard algebra, from (2.5) we obtain

$$q_n^{(\alpha,\beta)} = 2^{-n}(\alpha-\beta)\binom{2n+\alpha+\beta-1}{n-1},\ n=1,2,3,\dots. \tag{2.6}$$

2.2 In the case C_2) the polynomials $(n!)^{-1}W_n(z)$, $n=0,1,2,\dots$ are called Laguerre polynomials with parameter α. Usually, they are denoted by $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$.

The above definition yields

$$L_n^{(\alpha)}(z) = \frac{1}{n!z^{\alpha}\exp(-z)}\frac{d^n}{dz^n}\{z^{n+\alpha}\exp(-z)\}, \tag{2.7}$$

$$z\in\mathbb{C}\setminus(-\infty,0],\ n=0,1,2,\dots,$$

and by means of (1.10) we arrive to the representation

$$L_n^{(\alpha)}(z) = \sum_{k=0}^{n}\frac{(-1)^k}{k!}\binom{n+\alpha}{n-k}z^k,\ n=0,1,2,\dots. \tag{2.8}$$

Let $l_n^{(\alpha)},\ n=0,1,2,\dots$, and $m_n^{(\alpha)},\ n=1,2,3,\dots$, be the coefficient of z^n and z^{n-1}, respectively, in the n-th Laguerre polynomial with parameter α. Then from (2.8) it follows that

$$l_n^{(\alpha)} = \frac{(-1)^n}{n!},\ n=0,1,2,\dots, \tag{2.9}$$

and

$$m_n^{(\alpha)} = \frac{(-1)^{n-1}(n+\alpha)}{(n-1)!},\ n=1,2,3,\dots. \tag{2.10}$$

2.3 In the case C_3) the polynomials $(-1)^nW_n(z)$, $n=0,1,2,\dots$ are called Hermite polynomials and are denoted by $\{H_n(z)\}_{n=0}^{\infty}$. From this definition it follows that

$$H_n(z) = \frac{(-1)^n}{\exp(-z^2)} \frac{d^n}{dz^n}\{\exp(-z^2)\},\ n = 0, 1, 2, \ldots, \tag{2.11}$$

and (1.13) leads to the representation

$$H_n(z) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k (2z)^{n-2k}}{k!(n-2k)!},\ n = 0, 1, 2, \ldots. \tag{2.12}$$

If we denote by $h_n,\ n = 0, 1, 2, \ldots$, the coefficient of z^n in H_n, then from (2.12) we obtain

$$h_n = 2^n,\ n = 0, 1, 2, \ldots. \tag{2.13}$$

3. Orthogonality. Recurrence relations

3.1 The orthogonality of Jacobi polynomials in the "classical" case, i.e. when $\alpha > -1$ and $\beta > -1$ or, more generally, when $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\beta > -1$, is given by the following assertion:

(I.3.1) *If* $\operatorname{Re}\alpha > 1$ *and* $\operatorname{Re}\beta > 1$, *then*

$$\int_{-1}^{1} (1-t)^\alpha (1+t)^\beta P_m^{(\alpha,\beta)}(t) P_n^{(\alpha,\beta)}(t)\, dt = I_n^{(\alpha,\beta)} \delta_{mn},\ m, n = 0, 1, 2, \ldots, \tag{3.1}$$

where

$$I_n^{(\alpha,\beta)} = \begin{cases} \dfrac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & \text{if } n = 0; \\ \dfrac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}, & \text{if } n \geq 1. \end{cases} \tag{3.2}$$

Proof. For $n = 0, 1, 2, \ldots$ and $k = 0, 1, 2, \ldots, n$ we define

$$P_{n,k}^{(\alpha,\beta)} = \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta P_n^{(\alpha,\beta)}(t) t^k\, dt.$$

The substitution of the n-th Jacobi polynomial in the integrand by (2.1), followed by integrating by parts leads to

$$P_{n,k}^{(\alpha,\beta)} = \frac{(-1)^{n+k} k!}{n! 2^n} \int_{-1}^{1} \frac{d^{n-k}}{dt^{n-k}} \{(1-t)^{n+\alpha}(1+t)^{n+\beta}\}\, dt. \tag{3.3}$$

Since $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\beta > -1$, we have

$$\lim_{t\to\pm1\mp0} \frac{d^{n-k-1}}{dt^{n-k-1}} \{(1-t)^{n+\alpha}(1+t)^{n+\beta}\} = 0,\ n = 1, 2, 3, \ldots,$$

for each $k = 0, 1, 2, \dots, n-1$. Then from (1.3) it follows that

$$P_{n,k}^{(\alpha,\beta)} = \frac{(-1)^{n+k} k!}{n! 2^n} \int_{-1}^{1} \frac{d}{dt} \Big\{ \frac{d^{n-k-1}}{dt^{n-k-1}} [(1-t)^{n+\alpha}(1+t)^{n+\beta}] \Big\}\, dt = 0$$

whenever $n \geq 1$ and $k = 0, 1, 2, \dots, n-1$.

Since the system of monomials $\{1, t, t^2, \dots, t^{n-1}\},\ n \geq 1$ is a basis in the space of polynomials of degree not greater than $n-1$, the n-th Jacobi polynomial is orthogonal on the interval $(-1, 1)$ to any polynomial of degree less than n with respect to the weight function $w(t) = (1-t)^\alpha (1+t)^\beta$. Thus, the equalities (3.2) are proved when $m \neq n,\ m, n = 0, 1, 2, \dots$.

From (3.3) it follows that for $n = 0, 1, 2, \dots$

$$P_{n,n}^{(\alpha,\beta)} = 2^{-n} \int_{-1}^{1} (1-t)^{n+\alpha}(1+t)^{n+\beta}\, dt.$$

Substituting $t = 2u - 1$, we obtain

$$P_{n,n}^{(\alpha,\beta)} = 2^{n+\alpha+\beta+1} \int_{0}^{1} u^{n+\beta}(1-u)^{n+\alpha}\, du$$

$$= 2^{n+\alpha+\beta+1} B(n+\beta+1, n+\alpha+1) = \frac{2^{n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)}.$$

On the other hand, $P_n^{(\alpha,\beta)}(t) = p_n^{(\alpha,\beta)} t^n + R_{n-1}^{(\alpha,\beta)}(t)$, where $R_{n-1}^{(\alpha,\beta)}$ is a polynomial of degree less than n provided $n \geq 1$ and $R_{n-1}^{(\alpha,\beta)} \equiv 0$ when $n = 0$. Then, having in mind (2.3), we find that

$$\int_{-1}^{1} (1-t)^\alpha (1+t)^\beta \{P_n^{(\alpha,\beta)}(t)\}^2\, dt$$

$$= \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta P_n^{(\alpha,\beta)}(t) \{p_n^{(\alpha,\beta)} t^n + R_{n-1}^{(\alpha,\beta)}(t)\}\, dt$$

$$= p_n^{(\alpha,\beta)} \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta P_n^{(\alpha,\beta)}(t) t^n\, dt = p_n^{(\alpha,\beta)} P_{n,n}^{(\alpha,\beta)}$$

$$= \binom{2n+\alpha+\beta}{n} \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)} = I_n^{(\alpha,\beta)}$$

which confirms the validity of (3.1) also for the case $m = n,\ m, n = 0, 1, 2, \dots$.

3.2 The Jacobi polynomials have the property of orthogonality also "outside" the segment $[-1, 1]$ even in the case when one of the requirements of **(I.3.1)** is not satisfied. More precisely, the following assertion holds:

(I.3.2) *Suppose that* $\alpha+1, \beta+1$ *and* $\alpha+\beta+2$ *are not equal to* $0, -1, -2, \ldots$. *Then there exists a complex-valued function* $\varphi(\alpha, \beta; z)$, *holomorphic in the region* $\mathbb{C} \setminus [-1,1]$, *such that*

$$\int_\gamma \varphi(\alpha,\beta;z) P_m^{(\alpha,\beta)}(z) P_n^{(\alpha,\beta)}(z)\,dz \tag{3.4}$$

$$= 2\pi i I_n^{(\alpha,\beta)} \operatorname{ind}(\gamma; [-1,1])\delta_{mn},\ m,n = 0,1,2,\ldots.$$

for each closed rectifiable curve $\gamma \subset \mathbb{C} \setminus [1,1]$.

Proof. Let $\operatorname{Re}\alpha > -1, \operatorname{Re}\beta > -1$ and define

$$\varphi(\alpha,\beta;z) = -\int_{-1}^{1} \frac{(1-t)^\alpha (1+t)^\beta}{t-z}\,dt,\ z \in \mathbb{C} \setminus [-1,1]. \tag{3.5}$$

Then, the generalized Cauchy formula yields

$$\int_\gamma \varphi(\alpha,\beta;z) P_m^{(\alpha,\beta)}(z) P_n^{(\alpha,\beta)}(z)\,dz$$

$$= -\int_\gamma P_m^{(\alpha,\beta)}(z) P_n^{(\alpha,\beta)}(z)\,dz \int_{-1}^{1} \frac{(1-t)^\alpha(1+t)^\beta}{t-z}\,dt$$

$$= \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta\,dt \int_\gamma \frac{P_m^{(\alpha,\beta)}(z) P_n^{(\alpha,\beta)}(z)}{z-t}\,dz$$

$$= \operatorname{ind}(\gamma;t) 2\pi i \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta P_m^{(\alpha,\beta)}(t) P_n^{(\alpha,\beta)}(t)\,dt.$$

Since $\operatorname{ind}(\gamma;t)$=$\operatorname{ind}(\gamma;[-1,1])$ when $t \in [-1,1]$, the equalities (3.4) simply occur as corollaries of (3.1) in the case under consideration.

For each $\nu = 1,2,3,\ldots$ the function $\varphi_\nu(\alpha,\beta)$ defined by

$$\varphi_\nu(\alpha,\beta;z) = -\int_{-1+1/\nu}^{1-1/\nu} \frac{(1-t)^\alpha(1+t)^\beta}{t-z}\,dt$$

$$= -\int_{-1+1/\nu}^{1-1/\nu} \frac{\exp\{\alpha\log(1-t) + \beta\log(1+t)\}}{t-z}\,dt$$

is holomorphic as a function of the complex variables $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$ and $z \in \mathbb{C} \setminus [-1,1]$. Moreover, $\lim_{\nu\to\infty} \varphi_\nu(\alpha,\beta;z) = \varphi(\alpha,\beta;z)$ uniformly on each compact subset of the subregion of $\mathbb{C}^3$ defined by the inequalities $\operatorname{Re}\alpha > -1, \operatorname{Re}\beta > -1$ and the requirement $z \in \mathbb{C} \setminus [-1,1]$. Therefore, $\varphi(\alpha,\beta;z)$ is holomorphic in this subregion as a function of the complex variables α, β and z.

From the identity

$$\frac{1}{(1-t^2)(t-z)} = \frac{1}{1-z^2}\Big\{\frac{t+z}{1-t^2} + \frac{1}{t-z}\Big\},$$

considered for $t \in (-1,1)$ and $z \in \mathbb{C} \setminus [-1,1]$, we get

$$\varphi(\alpha,\beta;z) = -\int_{-1}^{1} \frac{(1-t)^{\alpha+1}(1+t)^{\beta+1}}{(1-t^2)(t-z)}\,dt = -\frac{1}{1-z^2}\Big\{\int_{-1}^{1} t(1-t)^{\alpha}(1+t)^{\beta}\,dt$$

$$+z\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}\,dt - \int_{-1}^{1}\frac{(1-t)^{\alpha+1}(1+t)^{\beta+1}}{t-z}\,dt\Big\}.$$

But

$$\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}\,dt = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}$$

and

$$\int_{-1}^{1} t(1-t)^{\alpha}(1+t)^{\beta}\,dt = \int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta+1}\,dt$$

$$-\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}\,dt = \frac{2^{\alpha+\beta+2}\Gamma(\alpha+1)\Gamma(\beta+2)}{\Gamma(\alpha+\beta+3)}$$

$$-\frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)(\beta-\alpha)}{\Gamma(\alpha+\beta+3)}.$$

Hence,

$$(1-z^2)\varphi(\alpha,\beta;z) \tag{3.6}$$

$$= \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+3)}[\beta-\alpha+(\alpha+\beta+2)z] + \varphi(\alpha+1,\beta+1;z)\Big\}.$$

The above relation was proved for $z \in \mathbb{C} \setminus [-1,1]$ provided $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\beta > -1$. It gives the analytical continuation of the function $\varphi(\alpha,\beta;z)$, as a holomorphic function of the variables α, β and z, in the region $(\mathbb{C} \setminus \mathbb{Z}^-) \times (\mathbb{C} \setminus \mathbb{Z}^-) \times (\mathbb{C} \setminus [-1,1])$. We denote this analytical continuation again by $\varphi(\alpha,\beta;z)$.

If m, n are fixed, then the both sides of equality (3.4) are holomorphic functions of the variables α and β in the region $B = (\mathbb{C} \setminus \mathbb{Z}^-) \times (\mathbb{C} \setminus \mathbb{Z}^-)$. They coincide in the subregion of B defined by the inequalities $\operatorname{Re}\alpha > -1, \operatorname{Re}\beta > -1$. According to the identity theorem, they coincide in the whole region B.

3.3 The orthogonality of the Laguerre polynomials when $\operatorname{Re}\alpha > -1$ is given by the following proposition:

(I.3.3). *If* $\operatorname{Re}\alpha > -1$, *then*

$$\int_0^{\infty} t^{\alpha}\exp(-t)L_m^{(\alpha)}(t)L_n^{(\alpha)}(t)\,dt = I_n^{(\alpha)}\delta_{mn},\ \ m,n = 0,1,2,\dots, \tag{3.7}$$

with

$$I_n^{(\alpha)} = \frac{\Gamma(n+\alpha+1)}{n!}, \ n = 0, 1, 2, \dots. \tag{3.8}$$

Proof. Define

$$L_{n,k}^{(\alpha)} = \int_0^\infty t^\alpha \exp(-t) L_n^{(\alpha)}(t) t^k \, dt, \ n = 0, 1, 2, \dots; k = 0, 1, 2, \dots, n. \tag{3.9}$$

From (2.7) it follows that

$$L_{n,k}^{(\alpha)} = \frac{(-1)^k k!}{n!} \int_0^\infty \frac{d^{n-k}}{dt^{n-k}} \{t^{n+\alpha} \exp(-t)\} \, dt. \tag{3.10}$$

Since $\operatorname{Re}\alpha > -1$, we have

$$\lim_{t\to+0} \frac{d^{n-k-1}}{dt^{n-k-1}} \{t^{n+\alpha} \exp(-t)\} \, dt = \lim_{t\to\infty} \frac{d^{n-k-1}}{dt^{n-k-1}} \{t^{n+\alpha} \exp(-t)\} \, dt = 0,$$

$$n \geq 1, \ k = 0, 1, 2, \dots, n-1,$$

and, hence,

$$L_{n,k}^{(\alpha)} = \frac{(-1)^k k!}{n!} \int_0^\infty \frac{d}{dt} \left\{ \frac{d^{n-k-1}}{dt^{n-k-1}} [t^{n+\alpha} \exp(-t)] \right\} dt = 0$$

when $n \geq 1$ and $k = 0, 1, 2, \dots, n-1$.

Thus, the validity of equalities (3.7) is established for $m \neq n$ since it turns out that for each $n \geq 1$ the n-th Laguerre polynomial is orthogonal on the interval $(0, \infty)$ to any polynomial of degree less than n with respect to the weight function $w(t) = t^\alpha \exp(-t)$.

The representation (3.10) yields

$$L_{n,n}^{(\alpha)} = (-1)^n \int_0^\infty t^{n+\alpha} \exp(-t) \, dt = (-1)^n \Gamma(n+\alpha+1), \ n = 0, 1, 2, \dots.$$

Since $L_n^{(\alpha)}(t) = l_n^{(\alpha)} t^n + R_{n-1}^{(\alpha)}(t)$, where $R_{n-1}^{(\alpha)}$ is a polynomial of deg $\leq n-1$ if $n \geq 1$ and $R_{n-1}^{(\alpha)} \equiv 0$ if $n = 0$, it follows that

$$\int_0^\infty t^\alpha \exp(-t) \{l_n^{(\alpha)}(t)\}^2 \, dt = l_n^{(\alpha)} \int_0^\infty t^\alpha \exp(-t) L_n^{(\alpha)}(t) t^n \, dt$$

$$= l_n^{(\alpha)} L_{n,n}^{(\alpha)} = \frac{\Gamma(n+\alpha+1)}{n!}, \ n = 0, 1, 2, \dots.$$

3.4 Let $L(\rho), 0 < \rho < \infty$, be the positively oriented loop around the nonnegative real semiaxis consisting of the ray $l'(\rho) : z = -t + i\rho, -\infty < t \leq 0$, of the semicircle $\delta(\rho) : z = \rho \exp i\theta, \pi/2 \leq \theta \leq 3\pi/2$, and of the ray $l''(\rho) : z = t - i\rho$, $0 \leq t < \infty$.

(I.3.4) *For* $\alpha \in \mathbb{C}, \rho \in (0, \infty)$ *and* $m, n = 0, 1, 2, \ldots,$

$$\int_{L(\rho)} (-z)^\alpha \exp(-z) L_m^{(\alpha)}(z) L_n^{(\alpha)}(z)\, dz = 2i \sin \alpha\pi . I_n^{(\alpha)} \delta_{mn}. \tag{3.11}$$

Proof. If α, m and n are fixed, then the integral in the left-hand side of (3.11) does not depend on ρ. In order to prove this assertion, we denote by $L(\rho, R)$, $0 < R < \infty$, the part of the loop $L(\rho)$ located in the half-plane $\operatorname{Re} z \leq R$. If $0 < \rho_1 < \rho_2$ and ν is a nonnegative integer, then by the Cauchy theorem

$$\int_{L(\rho_2,R)} (-z)^\alpha \exp(-z) z^\nu\, dz - \int_{L(\rho_1,R)} (-z)^\alpha \exp(-z) z^\nu\, dz$$

$$+ \int_{[R-i\rho_2,R-i\rho_1]} (-z)^\alpha \exp(-z) z^\nu\, dz + \int_{[R+i\rho_1,R+i\rho_2]} (-z)^\alpha \exp(-z) z^\nu\, dz = 0$$

Since

$$\lim_{R\to\infty} \int_{[R-i\rho_2,R-i\rho_1]} (-z)^\alpha \exp(-z) z^\nu\, dz = 0$$

and

$$\lim_{R\to\infty} \int_{[R+i\rho_1,R+i\rho_2]} (-z)^\alpha \exp(-z) z^\nu\, dz = 0,$$

we find that

$$\int_{L(\rho_1)} (-z)^\alpha \exp(-z) z^\nu\, dz = \int_{L(\rho_2)} (-z)^\alpha \exp(-z) z^\nu\, dz$$

If $\operatorname{Re} \alpha > -1$, then since $\lim_{\rho\to 0}(-t \pm i\rho)^\alpha = t^\alpha \exp(\pm i\alpha\pi), 0 < t < \infty$, and $\lim_{\rho\to 0} \int_{\delta(\rho)} (-z)^\alpha \exp(-z) z^\nu\, dz = 0$ for $\nu = 0, 1, 2, \ldots$, it follows that

$$\lim_{\rho\to 0} \int_{L(\rho)} (-z)^\alpha \exp(-z) L_m^{(\alpha)}(z) L_n^{(\alpha)}(z)\, dz$$

$$= 2i \sin \alpha\pi \int_0^\infty t^\alpha \exp(-t) L_m^{(\alpha)}(t) L_n^{(\alpha)}(t)\, dt$$

$$= 2i \sin \alpha\pi . I_n^{(\alpha)} \delta_{mn},\ \ m, n = 0, 1, 2, \ldots.$$

So far the validity of (3.11) was established only for the case $\operatorname{Re} \alpha > -1$. But if ρ, m and n are fixed, then both sides of (3.11) are holomorphic in the whole complex plane as functions of α. By the identity theorem, the equalities (3.11) hold for each $\alpha \in \mathbb{C}$.

As a consequence of the above assertion we arrive to the following conclusions:

(a) Let α be not an integer. In this case $\sin \alpha\pi \neq 0$ and the equalities (3.10) constitute the orthogonality of Laguerre's polynomials with parameter α "outside" the ray $[0, \infty)$.

(b) Let k be a positive integer. Since

$$\Gamma(z) = \frac{(-1)k - n - 1}{(k-n-1)!(z+k-n-1)} + h_{k,n}(z), \; n = 0, 1, 2, \ldots, k-1,$$

where $h_{k,n}$ is a function which is holomorphic in the neighbourhood of the point $-(k-n-1)$, we find that

$$\lim_{\alpha \to -k} \sin \alpha\pi\Gamma(n+\alpha+1) = (-1)^n \lim_{\alpha \to -k} \sin(n+\alpha+1)\pi\Gamma(n+\alpha+1)$$

$$= \frac{(-1)^n \pi}{(k-n-1)!}, \; n = 0, 1, 2, \ldots, k-1.$$

Therefore, for each $\rho \in (0, \infty)$ we have

$$\int_{L(\rho)} (-z)^\alpha \exp(-z) L_m^{(\alpha)}(z) L_n^{(\alpha)}(z)\, dz$$

$$= \begin{cases} \dfrac{2i(-1)^n \pi}{n!(k-n-1)!}\delta_{mn}, & \text{if } n = 0, 1, 2, \ldots, k-1; \\ 0, & \text{if } n = k, k+1, k+2, \ldots. \end{cases}$$

(c) Let α be a nonnegative integer, i.e. $\alpha = k, \; k = 0, 1, 2, \ldots$. In this case the equalities (3.11), i.e. the equalities

$$\int_{L(\rho)} (-z)^k \exp(-z) L_m^{(\alpha)}(z) L_n^{(\alpha)}(z)\, dz = 0, \; m, n = 0, 1, 2, \ldots,$$

follow immediately from the Cauchy theorem. Indeed,

$$\int_{L(\rho,R)} (-z)^k \exp(-z) L_m^{(\alpha)}(z) L_n^{(\alpha)}(z)\, dz$$

$$+ \int_{[R-i\rho, R+i\rho]} (-z)^k \exp(-z) L_m^{(\alpha)}(z) L_n^{(\alpha)}(z)\, dz = 0, \; m, n = 0, 1, 2, \ldots$$

and, moreover,

$$\lim_{R \to \infty} \int_{[R-i\rho, R+i\rho]} (-z)^k L_m^{(\alpha)}(z) L_n^{(\alpha)}(z)\, dz = 0, \; m, n = 0, 1, 2, \ldots.$$

The last cases **(b)** and **(c)** show that the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ are not orthogonal outside the ray $[0,\infty)$ with respect to the weight function $(-z)^{\alpha}\exp(-z)$ when α is an integer. But the next assertion shows that these polynomials are still orthogonal, but with respect to another weight function.

(I.3.5) *If* $k = 0, 1, 2, \dots$ *and* $\rho \in (0,\infty)$, *then*

$$\int_{L(\rho)} (-z)^k \exp(-z) \log(-z) L_m^{(\alpha)}(z) L_n^{(\alpha)}(z)\, dz \tag{3.12}$$

$$= 2\pi i(-1)^k I_n(k)\delta_{mn},\ m,n = 0,1,2,\dots.$$

The proof is analogous to that of **(I.3.4)**. It is based on the equality $\lim\limits_{\rho\to 0}\{\log(-t+i\rho)-\log(-t-i\rho)\} = 2\pi i, 0 < t\infty$, as well as on the fact that

$$\lim_{\rho\to 0}\int_{\delta(\rho)} (-z)^k \exp(-z)\log(-z) z^{\nu}\, dz = 0,\ k,\nu = 0,1,2,\dots.$$

3.5 For $\lambda \in (0,\infty)$ we denote by $p(\lambda)$ the image of the line $l(\lambda) : \zeta = -\xi$ $+i\lambda, -\infty < \xi < \infty$, by the map $z = \zeta^2$. It is easy to see that $p(\lambda)$ is the parabola with Cartesian equation $y^2 = 4\lambda^2(x+\lambda^2)$, i.e. the parabola with vertex at the point $(-\lambda^2, 0)$ and focus at the origin.

(I.3.6) *If* $\lambda \in (0,\infty)$ *and* α *is not an integer, then the Laguerre polynomials with parameter* α *are orthogonal on the parabola* $p(\lambda)$ *with respect to the weight function* $(-z)^{\alpha}\exp(-z)$, *i.e*

$$\int_{p(\lambda)} (-z)^{\alpha} \exp(-z) L_m^{(\alpha)}(z) L_n^{(\alpha)}(z)\, dz \tag{3.13}$$

$$= 2i\sin\alpha\pi . I_n^{(\alpha)}\delta_{mn}\, m,n = 0,1,2,\dots.$$

In particular, for $k = 0,1,2,\dots,$

$$\int_{p(\lambda)} (-z)^k \exp(-z)\log(-z) L_m^{(k)}(z) L_n^{(k)}(z)\, dz \tag{3.14}$$

$$= 2\pi i(-1)^k I_n^{(k)}\delta_{mn},\ m,n = 0,1,2,\dots.$$

Proof. Let $p(\lambda, R)$ be the part of $p(\lambda)$ lying in the half-plane $\operatorname{Re} z \le R$. If $0 < \rho < \lambda^2$ and $R \pm i\sigma(\lambda, R)$ are the points of intersection of $p(\lambda)$ and the line $\operatorname{Re} z = R$, then the Cauchy theorem gives that for $\nu = 0,1,2,\dots,$

$$\int_{p(\lambda,R)} (-z)^{\alpha}\exp(-z) z^{\nu}\, dz - \int_{L(\rho,R)} (-z)^{\alpha}\exp(-z) z^{\nu}\, dz \tag{3.15}$$

$$= -\int_{[R-i\sigma(\lambda,R),R-i\rho]} (-z)^{\alpha} \exp(-z) z^{\nu}\, dz - \int_{[R+i\rho,R+i\sigma(\lambda,R)]} (-z)^{\alpha} \exp(-z) z^{\nu}\, dz.$$

Since $\sigma(\lambda, R) = 2\lambda\sqrt{R+\lambda^2}$, there exist real constants $A = A(\alpha, \lambda)$ and a $=a(\alpha, \nu)$ such that

$$\left| \int_{[R+i\rho,R+i\sigma(\lambda,R)]} (-z)^{\alpha} \exp(-z) z^{\nu}\, dz \right| \le AR^{a} \exp(-R)$$

and

$$\left| \int_{[r+i\rho,R+i\sigma(\lambda,R)]} (-z)^{\alpha} \exp(-z) z^{\nu}\, dz \right| \le AR^{a} \exp(-R).$$

Then from (3.15) it follows that

$$\int_{p(\lambda)} (-z)^{\alpha} \exp(-z) z^{\nu}\, dz = \int_{L(\rho)} (-z)^{\alpha} \exp(-z) z^{\nu}\, dz$$

for $\nu = 0, 1, 2, \ldots$ and in order to finish the proof, we refer to **(I.3.4)**. In a similar way we can prove the validity of the equalities (3.14).

3.6 The orthogonality of the Hermite polynomials is usually expressed by the following proposition:

(I.3.7) *The equalities*

$$\int_{-\infty}^{\infty} \exp(-t^2) H_m(t) H_n(t)\, dt = I_n \delta_{mn},\ m, n = 0, 1, 2, \ldots, \tag{3.16}$$

hold with

$$I_n = n! 2^n \sqrt{\pi},\ n = 0, 1, 2, \ldots. \tag{3.17}$$

Proof. Define

$$H_{n,k} = \int_{-\infty}^{\infty} \exp(-t^2) H_n(t) t^k\, dt,\ n, k = 0, 1, 2, \ldots.$$

From [Chapter I, (2.11)] it follows that if $n \ge 1$ and $k = 0, 1, 2, \ldots, n-1$, then

$$H_{n,k} = (-1)^{n+k} k! \int_{-\infty}^{\infty} \frac{d^{n-k}}{dt^{n-k}} \{\exp(-t^2)\}\, dt$$

$$= (-1)^{n+k} k! \int_{-\infty}^{\infty} \frac{d}{dt} \left\{ \frac{d^{n-k-1}}{dt^{n-k-1}} [\exp(-t^2)] \right\} dt$$

$$= -k! \int_{-\infty}^{\infty} \frac{d}{dt} \{\exp(-t^2) H_{n-k-1}(t)\}\, dt.$$

Since $\lim_{t\to\pm\infty}\exp(-t^2)t^\nu = 0$ for $\nu = 0,1,2,\dots$, we get $H_{n,k} = 0$ whenever $n \geq 1$ and $k = 0,1,2,\dots,n-1$. This confirms the equalities (3.16) when $m \neq n,\ m,n = 0,1,2,\dots$. In particular,

$$H_{n,n} = n!\int_{-\infty}^{\infty}\exp(-t^2)\,dt = n!\int_0^{\infty}u^{-1/2}\exp(-u)\,du$$

$$= n!\Gamma(1/2) = n!\sqrt{\pi},\ n = 0,1,2,\dots,$$

and, hence,

$$I_n = \int_{-\infty}^{\infty}\exp(-t^2)\{H_n(t)\}^2\,dt = h_n n!\sqrt{\pi} = n!2^n\sqrt{\pi},\ n = 0,1,2,\dots.$$

We leave the proof of the following assertion as an exercise to the reader:

(I.3.8) *For $m,n = 0,1,2,\dots,$ the equalities*

$$\int_{l(\tau)}\exp(-z^2)H_m(z)H_n(z)\,dz = n!2^n\sqrt{\pi}\delta_{mn},\ m,n = 0,1,2,\dots. \tag{3.18}$$

hold, where $l(\tau)$ is the line $z = t + i\tau, -\infty < t < \infty, \tau \in \mathbb{R}$.

3.7 Denote by $\mathcal{P}(\mathbb{C})$ the set of all polynomials with complex coefficients and suppose that $\mathcal{B}(X,Y)$ is a bilinear form on $\mathcal{P}(\mathbb{C})\times\mathcal{P}(\mathbb{C})$. A system of polynomials $\{X_n(z)\}_{n=0}^{\infty} \subset \mathcal{P}(\mathbb{C})$ such that $\deg X_n = n,\ n = 0,1,2,\dots,$ is called orthogonal with respect to the form $\mathcal{B}$ if $\mathcal{B}(X_m,X_n) = A_n\delta_{mn}, m,n = 0,1,2,\dots,$ and $A_n \neq 0$ for each $n = 0,1,2,\dots$.

Remark. It is clear that in order to exists a polynomial system orthogonal to a bilinear form, it is necessary this form to be nontrivial.

(I.3.9) *If a system of polynomials $\{X_n(z)\}_{n=0}^{\infty}$ such that $\deg X_n = n,\ n = 0,1,2,\dots,$ is orthogonal with respect to a nontrivial bilinear form $\mathcal{B}(X,Y)$ with the property $\mathcal{B}(zX(z),Y(z)) = \mathcal{B}(X(z),zY(z)),\ X,Y \in \mathcal{P}(\mathbb{C})$, then it is a solution of a second order linear recurrence equation of the kind*

$$a_n y_{n+1} + (z-\lambda_n)y_n + c_n y_{n-1} = 0, \tag{3.19}$$

where $a_n \neq 0$ and $c_n \neq 0,\ n = 1,2,3,\dots$.

Proof. Since $\deg X_n = n,\ n = 0,1,2,\dots,$ the system $\{X_n(z)\}_{n=0}^{\infty}$ is linearly independent and, hence, it is a basis of $\mathcal{P}(\mathbb{C})$ considered as a $\mathbb{C}$-vector space. In particular,

$$zX_n(z) = \sum_{k=0}^{n+1}\lambda_{n,k}X_{n+1-k}(z),\ n = 0,1,2,\dots. \tag{3.20}$$

From the orthogonality with respect to the form $\mathcal{B}$ it follows that if $n \geq 2$ and $\nu = 0, 1, 2, \ldots, n-2$, then $\mathcal{B}(zX_n(z), X_\nu(z)) = \sum_{k=0}^{n+1} \lambda_{n,k} \mathcal{B}(X_{n+1-k}(z), X_\nu(z))$ $= \lambda_{n,n+1-\nu} A_\nu$.

Since

$$zX_\nu(z) = \sum_{s=0}^{\nu+1} \mu_{\nu,s} X_s(z), \tag{3.21}$$

we obtain

$$\mathcal{B}(zX_n(z), X_\nu(z)) = \mathcal{B}(X_n(z), zX_\nu(z)) = \sum_{s=0}^{\nu+1} \mu_{\nu,s} \mathcal{B}(X_n(z), X_s(z)) = 0$$

for $\nu = 0, 1, 2, \ldots, n-2$. Therefore, $\lambda_{n,n+1-\nu} = 0$ if $\nu = 0, 1, 2, \ldots, n-2$, i.e., relation (3.20) has in fact the form $zX_n(z) = \lambda_{n,0} X_{n+1}(z) + \lambda_{n,1} X_n(z) + \lambda_{n,2} X_{n-1}(z)$. If we set $a_n = -\lambda_{n,0}, \lambda_n = \lambda_{n,1}$ and $c_n = -\lambda_{n,2},\ n = 1, 2, 3, \ldots$, then the last relation takes the form

$$a_n X_{n+1}(z) + (z - \lambda_n) X_n(z) + c_n X_{n-1}(z) = 0,\ n = 1, 2, 3, \ldots. \tag{3.22}$$

All the coefficients of the polynomial in the left-hand side in (3.22) are equal to zero. If $X_\nu(z) = \sum_{k=0}^{\nu} p_{\nu,k} z^{\nu-k},\ \nu = 0, 1, 2, \ldots$, then, in particular, $a_n p_{n+1,0}$ $+ p_{n,0} = 0$ and $a_n p_{n+1,1} - \lambda_n p_{n,0} = 0$. Hence,

$$a_n = -\frac{p_{n,0}}{p_{n+1,0}}, \quad n = 1, 2, 3, \ldots \tag{3.23}$$

and

$$\lambda_n = \frac{p_{n,1}}{p_{n,0}} - \frac{p_{n+1,1}}{p_{n+1,0}},\ n = 1, 2, 3, \ldots. \tag{3.24}$$

From (3.21) with $\nu = n-1$ it follows that $\mu_{n-1,n} = p_{n-1,0}/p_{n,0}$. Then (3.21) with $\nu = n-1$ as well as (3.22) lead to $\mathcal{B}(zX_n(z), X_{n-1}(z)) = \mathcal{B}(X_n(z), zX_{n-1}(z))$ $= -c_n A_{n-1} = \mu_{n-1,n} A_n,\ n = 1, 2, 3, \ldots$, i.e.

$$c_n = -\frac{p_{n-1,0}}{p_{n,0}} \cdot \frac{A_n}{A_{n-1}},\ n = 1, 2, 3, \ldots. \tag{3.25}$$

3.8 Suppose that $\alpha+1, \beta+1$ and $\alpha+\beta+2$ are not equal to $0, -1, -2, \ldots$ and that $\gamma \subset \mathbb{C} \setminus [-1, 1]$ is a closed rectifiable curve with $\operatorname{ind}(\gamma; [-1, 1]) = 1$. Define

$$J^{(\alpha,\beta)}(X, Y) = \frac{1}{2\pi i} \int_\gamma \varphi(\alpha, \beta; z) X(z) Y(z)\, dz,\ X, Y \in \mathcal{P}(\mathbb{C}). \tag{3.26}$$

The system of the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ is orthogonal with respect to the bilinear form (3.26). Moreover, $J^{(\alpha,\beta)}(zX(z),Y(z)) = J^{(\alpha,\beta)}(X(z),zY(z))$ whenever $X, Y \in \mathcal{P}(\mathbb{C})$ and, hence, the system of Jacobi polynomials is a solution of a recurrence relation of the kind (3.22). The calculation of a_n, λ_n and c_n leads to the following proposition:

(I.3.10) *If $\alpha+1, \beta+1$ and $\alpha+\beta+2$ are not equal to $0, -1, -2, \dots,$ then the system of Jacobi polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ is a solution of the recurrence equation*

$$
\begin{aligned}
(3.27) \qquad & -\frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} y_{n+1} \\
& + \left(z - \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} \right) y_n \\
& - \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} y_{n-1} = 0.
\end{aligned}
$$

Remark. If $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\beta > -1$, then according to (3.5) we find that

$$J^{(\alpha,\beta)}(X,Y) = \int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta} X(t)Y(t)\,dt.$$

3.9 Suppose that $\alpha \in \mathbb{C}$ is not an integer. Define

$$L^{(\alpha)}(X,Y) = \frac{1}{2i\sin\alpha\pi} \int_{L(\rho)} (-z)^{\alpha} \exp(-z) X(z)Y(z)\,dz$$

as well as

$$L^{(k)}(X,Y) = \frac{(-1)^k}{2\pi i} \int_{L(\rho)} (-z)^k \exp(-z) \log(-z) X(z)Y(z)\,dz,\ \ k = 0, 1, 2, \dots.$$

Thus, for each $\alpha \in \mathbb{C}\setminus\mathbb{Z}^-$, a bilinear form $L^{(\alpha)}(X,Y)$ on $\mathcal{P}(\mathbb{C})\times\mathcal{P}(\mathbb{C})$ such that $L^{(\alpha)}(zX(z),Y(z)) = L^{(\alpha)}(X(z),zY(z))$ is defined. Since the system of Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ is orthogonal with respect to $L^{(\alpha)}$, it is a solution of a recurrence equation of the kind (3.19). The calculation of a_n, λ_n and c_n in this case leads to the following proposition:

(I.3.11) *If $\alpha \in \mathbb{C}\setminus\mathbb{Z}^-$, then the system of Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ is a solution of the recurrence equation*

$$(3.28) \qquad (n+1)y_{n+1} + (z - 2n - \alpha - 1)y_n + (n+\alpha)y_{n-1} = 0.$$

Remark. If $\mathrm{Re}\,\alpha > -1$, then

$$L^{(\alpha)}(X, Y) = \int_0^\infty t^\alpha \exp(-t) X(t) Y(t)\, dt.$$

3.10 The system of Hermite polynomials $\{H_n(z)\}_{n=0}^\infty$ is orthogonal with respect to the bilinear form H defined as

$$H(X, Y) = \int_{-\infty}^\infty \exp(-t^2) X(t) Y(t)\, dt.$$

Since $H(zX(z), Y(z)) = H(X(z), zY(z))$, whenever $X, Y \in \mathcal{P}(\mathbb{C})$, this system is a solution of a recurrence equation of the kind (3.19). By calculating a_n, λ_n and c_n we arrive to the following assertion:

(I.3.12) *The system of Hermite polynomials* $\{H_n(z)\}_{n=0}^\infty$ *is a solution of the recurrence equation*

$$(-1/2) y_{n+1} + z y_n - n y_{n-1} = 0. \tag{3.29}$$

4. Jacobi, Laguerre and Hermite associated functions. Christoffel-Darboux type formulas

4.1 Denote by $e(r)$ the image of the circle $C(0; r) : z = r \exp i\theta, 0 \le \theta \le 2\pi$, $r > 1$, by the Zhukovskii transformation, i.e., by the map $\zeta = (\omega + \omega^{-1})/2$. As it is well-known, $e(r)$ is the (positively oriented) ellipse with focuses at the points -1 and 1 and with semiaxes $(r + r^{-1})/2$ and $(r - r^{-1})/2$, respectively. Each ellipse of this kind is an image of a circle $C(0; r)$ with $r > 1$.

Suppose that $\alpha + 1, \beta + 1$ and $\alpha + \beta + 2$ are not equal to $0, -1, -2, \ldots$. Define for $z \in \mathbb{C} \setminus [-1, 1]$ and $n = 0, 1, 2, \ldots$

$$Q_n^{(\alpha,\beta)}(z) = \frac{1}{2\pi i} \int_{e(r)} \frac{\varphi(\alpha, \beta; \zeta) P_n^{(\alpha,\beta)}(\zeta)}{\zeta - z}\, d\zeta, \tag{4.1}$$

where $r > 1$ is chosen so that $z \in \mathrm{ext}\, e(r)$. These functions are called Jacobi associated functions.

The above definition implies that for $n = 0, 1, 2, \ldots$ the function $Q_n^{(\alpha\beta)}(z)$ is holomorphic in the region $\mathbb{C} \setminus [-1, 1]$. In fact, it is holomorphic in the subregion $\overline{\mathbb{C}} \setminus [-1, 1]$ of the extended complex plane and $Q_n^{(\alpha,\beta)}(\infty) = 0,\ n = 0, 1, 2, \ldots$.

Since the function $\varphi(\alpha, \beta; z)$ is holomorphic in the region $\overline{\mathbb{C}} \setminus [-1, 1]$, and $\varphi(\alpha, \beta; \infty) = 0$, the Cauchy formula gives

$$Q_0^{(\alpha,\beta)}(z) = -\frac{1}{2\pi i} \int_{e(r)} \frac{\varphi(\alpha, \beta; \zeta)}{\zeta - z}\, d\zeta = \varphi(\alpha, \beta; z). \tag{4.2}$$

Suppose that $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\beta > -1$. Then, as a result of (3.5),(4.1) and the identity

$$\frac{1}{(\zeta - z)(t-\zeta)} = (t-z)^{-1}\Big(\frac{1}{\zeta - z} + \frac{1}{t-\zeta}\Big),$$

it follows that for $z \in \mathbb{C} \setminus [-1,1]$,

$$Q_n^{(\alpha,\beta)}(z) = \frac{1}{2\pi i}\int_{e(r)} \frac{P_n^{(\alpha,\beta)}(\zeta)}{\zeta - z}\,d\zeta \int_{-1}^{1} \frac{(1-t)^\alpha(1+t)^\beta}{t-\zeta}\,dt$$

$$= \frac{1}{2\pi i}\int_{-1}^{1} \frac{(1-t)^\alpha(1+t)^\beta}{t-z}\int_{e(r)} P_n^{(\alpha,\beta)}(\zeta)\Big(\frac{1}{\zeta - z} + \frac{1}{t-\zeta}\Big)\,d\zeta,$$

provided $r > 1$ and z is outside $e(r)$. Since by the Cauchy theorem

$$\int_{e(r)} \frac{P_n^{(\alpha,\beta)}(\zeta)}{\zeta - z}\,d\zeta = 0,\ \ n = 0,1,2,\dots,$$

the Cauchy formula gives that

$$Q_n^{(\alpha,\beta)}(z) = -\int_{-1}^{1} \frac{(1-t)^\alpha(1+t)^\beta P_n^{(\alpha,\beta)}(t)}{t-z}\,dt, \quad n = 0,1,2,\dots. \tag{4.3}$$

Applying now (2.1) we find that

$$Q_n^{(\alpha,\beta)}(z) = \frac{(-1)^{n+1}}{n!2^n}\int_{-1}^{1} \frac{\{(1-t)^{n+\alpha}(1+t)^{n+\beta}\}^{(n)}}{t-z}\,dt, \quad n = 0,1,2,\dots,$$

and after integration by parts we obtain that for $z \in \mathbb{C} \setminus [-1,1]$ and $n = 0,1,2,\dots$,

$$Q_n^{(\alpha,\beta)}(z) = (-1)^{n+1}2^{-n}\int_{-1}^{1} \frac{(1-t)^{n+\alpha}(1+t)^{n+\beta}}{(t-z)^{n+1}}\,dt. \tag{4.4}$$

Remark. We recall that the above representation was established provided $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\beta > -1$.

We assume again that the complex numbers $\alpha+1, \beta+1$ and $\alpha+\beta+2$ are not equal $0,-1,-2,\dots$ and define for $n = 1,2,3,\dots$

$$a_n^{(\alpha,\beta)} = -\frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)},$$

$$\lambda_n^{(\alpha,\beta)} = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$

$$c_n^{(\alpha,\beta)} = -\frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}.$$

Then the recurrence equation (3.27) for Jacobi's polynomials leads to the relation

$$a_n^{(\alpha,\beta)} P_{n+1}^{(\alpha,\beta)}(z) + (z - \lambda_n^{(\alpha,\beta)}) P_n^{(\alpha,\beta)}(z) + c_n^{(\alpha,\beta)} P_{n-1}^{(\alpha,\beta)}(z) = 0, \tag{4.5}$$

which holds for $z \in \mathbb{C}$ and $n = 1, 2, 3, \ldots$. By using it as well as the definition (4.1) of Jacobi's associated functions, we find that for $z \in \mathbb{C} \setminus [-1, 1]$ and $n = 1, 2, 3, \ldots$, it holds

$$\begin{aligned} &a_n^{(\alpha,\beta)} Q_{n+1}^{(\alpha,\beta)}(z) + (z - \lambda_n^{(\alpha,\beta)}) Q_n^{(\alpha,\beta)}(z) + c_n^{(\alpha,\beta)} Q_{n-1}^{(\alpha,\beta)}(z) \\ &= -\frac{1}{2\pi i} \int_{e(r)} \varphi(\alpha, \beta; \zeta)(\zeta - z)^{-1} \{ a_n^{(\alpha,\beta)} P_{n+1}^{(\alpha,\beta)}(\zeta) + (\zeta - \lambda_n^{(\alpha,\beta)}) P_n^{(\alpha,\beta)}(\zeta) \\ &\quad + c_n^{(\alpha,\beta)} P_{n-1}^{(\alpha,\beta)}(\zeta) \} \, d\zeta = \frac{1}{2\pi i} \int_{e(r)} \varphi(\alpha, \beta; \zeta) P_n^{(\alpha,\beta)}(\zeta) \, d\zeta. \end{aligned}$$

Then the orthogonality of the Jacobi polynomials or more precisely the equalities (3.4) with $m = 0$ and $n = 1, 2, 3, \ldots$ yields

$$a_n^{(\alpha,\beta)} Q_{n+1}^{(\alpha,\beta)}(z) + (z - \lambda_n^{(\alpha,\beta)}) Q_n^{(\alpha,\beta)}(z) + c_n^{(\alpha\beta)} Q_{n-1}^{(\alpha,\beta)}(z) = 0, \tag{4.6}$$

for each $z \in \mathbb{C} \setminus [-1, 1]$ and $n = 1, 2, 3, \ldots$, i.e. the system $\{Q_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ of Jacobi associated functions is also a solution of the recurrence equation (3.27).

4.2 Suppose that $\alpha + 1 \notin \mathbb{Z}^- \cup \{0\}$. Then the functions $\{M_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ defined in the region $\mathbb{C} \setminus [0, \infty)$ as

$$M_n^{(\alpha)}(z) = -\frac{1}{2i \sin \alpha\pi} \int_{p(\lambda)} \frac{(-\zeta)^\alpha \exp(-\zeta) L_n^{(\alpha)}(\zeta)}{\zeta - z} \, d\zeta \tag{4.7}$$

when α is not an integer, and by the equalities

$$M_n^{(k)}(z) = -\frac{(-1)^k}{2pii} \int_{p(\lambda)} \frac{(-\zeta)^k \exp(-\zeta) \log(-\zeta) L_n^{(k)}(\zeta)}{\zeta - z} \, d\zeta \tag{4.8}$$

for $k = 0, 1, 2, \ldots$, where $\lambda \in (0, \infty)$ is chosen so that z is outside $p(\lambda)$, are called Laguerre associated functions.

From (4.7) and (4.8) it follows that if $\alpha \in \mathbb{C} \setminus \mathbb{Z}^-$, then for each $n = 0, 1, 2 \ldots$, $M_n^{(\alpha)}(z)$ is a holomorphic function in the region $\mathbb{C} \setminus [0, \infty)$.

Suppose that α is not an integer and that $\operatorname{Re} \alpha > -1$. Then (4.7) gives

$$M_n^{(\alpha)}(z) = -\frac{1}{2i \sin \alpha\pi} \lim_{\lambda \to 0} \int_{p(\lambda)} \frac{(-\zeta)^\alpha \exp(-\zeta) L_n^{(\alpha)}(\zeta)}{\zeta - z} \, d\zeta, \ n = 0, 1, 2, \ldots,$$

i.e.

$$M_n^{(\alpha)}(z) = -\int_0^\infty \frac{t^\alpha \exp(-t) L_n^{(\alpha)}(t)}{t - z} \, dt, \ n = 0, 1, 2, \ldots \tag{4.9}$$

Similarly, using (4.8), we can prove the validity of representation (4.9) when $\alpha = k$ is a nonnegative integer. Then, integrating by parts, as a corollary of (2.7) we find the following integral representation of the Laguerre associated functions in the region $\mathbb{C} \setminus [0, \infty)$:

$$M_n^{(\alpha)}(z) = -\int_0^\infty \frac{t^{n+\alpha}\exp(-t)}{(t-z)^{n+1}}\,dt,\ n = 0, 1, 2, \ldots. \tag{4.10}$$

Remark. We point out that the validity of (4.10) is proved under the assumption $\operatorname{Re}\alpha > -1$.

Let us put $a_n^{(\alpha)} = n+1, \lambda_n^{(\alpha)} = 2n+\alpha+1$ and $c_n^{(\alpha)} = n+\alpha,\ n = 1, 2, 3, \ldots.$ Then, recurrence equation (3.28) yields that for each $z \in \mathbb{C}$ and $n = 1, 2, 3, \ldots$, it holds

$$a_n^{(\alpha)} L_{n+1}^{(\alpha)}(z) + (z - \lambda_n^{(\alpha)}) L_n^{(\alpha)}(z) + c_n^{(\alpha)} L_{n-1}^{(\alpha)}(z) = 0. \tag{4.11}$$

The definition of the Laguerre associated functions by (4.7) and (4.8) as well as the orthogonality of the Laguerre polynomials [(**I.3.6**)] provide the possibility to conclude that for $z \in \mathbb{C} \setminus [0, \infty)$ and $n = 1, 2, 3, \ldots$, it holds

$$a_n^{(\alpha)} M_{n+1}^{(\alpha)}(z) + (z - \lambda_n^{(\alpha)}) M_n^{(\alpha)}(z) + c_n^{(\alpha)} M_{n-1}^{(\alpha)}(z) = 0. \tag{4.12}$$

Hence, for $z \in \mathbb{C} \setminus [0, \infty)$ the system $\{M_n^{(\alpha)}(z)\}_{n=0}^\infty$, where $\alpha \in \mathbb{C} \setminus \mathbb{Z}^-$, is a solution of the recurrence equation (3.28).

4.3 The functions $\{G_n(z)\}_{n=0}^\infty$, defined in the open set $\mathbb{C} \setminus \mathbb{R}$ by means of the equalities

$$G_n(z) = -\int_{-\infty}^\infty \frac{\exp(-t^2)H_n(t)}{t-z}\,dt,\ n = 0, 1, 2, \ldots, \tag{4.13}$$

are called Hermite associated functions.

Evidently, these functions are holomorphic in the open set $\mathbb{C} \setminus \mathbb{R}$. Further, using (2.11), (4.10) and integrating by parts, we get the following integral representation of the Hermite associated functions in $\mathbb{C} \setminus \mathbb{R}$:

$$G_n(z) = (-1)^{n+1} n! \int_{-\infty}^\infty \frac{\exp(-t^2)}{(t-z)^{n+1}}\,dt,\ n = 0, 1, 2, \ldots. \tag{4.14}$$

Using (4.13) and the orthogonality of the Hermite polynomials, one can easily prove that the system of associated Hermite functions is a solution of the recurrence equation (3.29). Indeed, if $z \in \mathbb{C} \setminus \mathbb{R}$ and $n = 1, 2, 3, \ldots$, then

$$(-1/2)G_{n+1}(z) + zG_n(z) - nG_{n-1}(z) \tag{4.15}$$

$$= -\int_{-\infty}^{\infty} \exp(-t^2)(t-z)^{-1}\{(-1/2)H_{n+1}(t) + (z-t)H_n(t) + tH_n(t) - nH_{n-1}(t)\}\,dt$$

$$= \int_{-\infty}^{\infty} \exp(-t^2)H_n(t)\,dt = 0.$$

4.4 Suppose again that $\mathcal{B}(X,Y)$ is a nontrivial bilinear form on the space $\mathcal{P}(\mathbb{C})$ of the polynomials with complex coefficients with the property that $\mathcal{B}(zX(z),Y(z)) = \mathcal{B}(X(z),zY(z))$ whenever $X,Y \in \mathcal{P}(\mathbb{C})$. It was proved that each system of polynomials $\{X_n(z)\}_{n=0}^{\infty}$, $\deg X_n = n$, $n = 0,1,2,\dots$, which is orthogonal with respect to such a form, is a solution of a recurrence relation of the kind (3.19) with $a_n \neq 0$ and $c_n \neq 0$, $n = 1,2,3,\dots$ [**(I.3.9)**].

Having in mind (3.23), we define $a_0 = -p_{0,0}/p_{1,0}$ and

$$k_n = \frac{a_n}{A_n},\ n = 0,1,2\dots. \tag{4.16}$$

Then from (3.25) it follows that

$$\frac{c_n}{A_n} = -\frac{p_{n-1,0}}{p_{n,0}A_{n-1}} = \frac{a_{n-1}}{A_{n-1}} = k_{n-1},\ n = 1,2,3,\dots.$$

Now, it is clear that equation (3.19) takes the form

$$k_n y_{n+1} + \frac{1}{A_n}(z-\lambda_n) + k_{n-1}y_{n-1} = 0, \tag{4.17}$$

which we name canonical. Likewise, equality (3.22) can be rewritten as

$$k_n X_{n+1}(z) + \frac{1}{A_n}(z-\lambda_n)X_n(z) + k_{n-1}X_{n-1}(z) = 0 \tag{4.18}$$

Suppose that the system of complex-valued functions $\{Y_n(z)\}_{n=0}^{\infty}$, defined in a nonempty subset D of $\mathbb{C}$, is also a solution of equation (3.19). Then the equality

$$k_n Y_{n+1}(\zeta) + \frac{1}{A_n}(\zeta-\lambda_n)Y_n(\zeta) + k_{n-1}Y_{n-1}(\zeta) = 0 \tag{4.19}$$

holds whenever $\zeta \in D$ and $n = 0,1,2,\dots$.

We define $\Omega_n(z,\zeta) = k_n\{X_n(z)Y_{n+1}(\zeta) - X_{n+1}(z)Y_n(\zeta)\}$ for $z \in \mathbb{C}, \zeta \in D$ and $n = 0,1,2,\dots$. Then from (4.18) and (4.19) it follows that

$$\Omega_n(z,\zeta) + \frac{1}{A_n}X_n(z)Y_n(\zeta) - \Omega_{n-1}(z,\zeta),\ n = 1,2,3,\dots. \tag{4.20}$$

If ν is a nonnegative integer, then (4.20) yields

$$\Omega_\nu(z,\zeta) + (\zeta - z)\sum_{n=0}^{\nu}\frac{1}{A_n}X_n(z)Y_n(\zeta) = \Omega(z,\zeta),$$

where

$$\Omega(z,\zeta)=\Omega_0(z,\zeta)+\frac{1}{A_0}(\zeta-z)X_0(z)Y_0(\zeta). \tag{4.21}$$

Therefore, if $z\in\mathbb{C},\zeta\in D$ and $z\neq\zeta$, then

$$\frac{\Omega(z,\zeta)}{\zeta-z}=\sum_{n=0}^{\nu}\frac{1}{A_n}X_n(z)Y_n(\zeta)+\frac{\Omega_\nu(z,\zeta)}{\zeta-z}. \tag{4.22}$$

Suppose that there exists an "extension" of the bilinear form $\mathcal{B}(X,Y)$, as a function of the second variable, to the linear hull of the polynomials with complex coefficients and the Cauchy kernels $(z-\zeta)^{-1}$ considered as functions of $z\in\mathbb{C}\setminus\{\zeta\}$, where ζ is an arbitrary complex number. Then we define

$$Y_n(\zeta)=-\mathcal{B}(X_n(w),(w-\zeta)^{-1}) \tag{4.23}$$

for $\zeta\in D,w\in\mathbb{C}\setminus\{\zeta\}$ and $n=0,1,2,\dots$

Suppose further that the property $\mathcal{B}(wX(w),Y(w))=\mathcal{B}(X(w),wY(w))$ still holds after substituting $Y(w)$ for $(w-\zeta)^{-1}$. Then from (4.20) it follows that

$$a_nY_{n+1}(\zeta)+(\zeta-\lambda_n)Y_n(\zeta)+c_nY_{n-1}(\zeta)$$

$$=-a_n\mathcal{B}(X_{n+1}(w),(w-\zeta)^{-1})-(\zeta-\lambda_n)\mathcal{B}(X_n(w),(w-\zeta)^{-1})$$
$$-c_n\mathcal{B}(X_{n-1}(w),(w-\zeta)^{-1})=-\mathcal{B}(a_nX_{n+1}(w)+(\zeta-\lambda_n)X_n(w)+c_nX_{n-1}(w),$$
$$(w-\zeta)^{-1})=-\mathcal{B}(a_nX_{n+1}(w)+(w-\lambda_n)X_n(w)+c_nX_{n-1}(w)+(\zeta-w)X_n(w),$$
$$(w-\zeta)^{-1})=-\mathcal{B}((\zeta-w)X_n(w),(w-\zeta^{-1}))=-\mathcal{B}(X_n(w),-1)=B(X_n(w),1)$$
$$=(p_{0,0})^{-1}\mathcal{B}(X_n(w),X_0(w))=0,\ n=1,2,3,\dots.$$

This means that the system of functions $\{Y_n(\zeta)\}_{n=0}^{\infty}$, defined by the equalities (4.23), is a solution of the recurrence equation $a_ny_{n+1}+(\zeta-\lambda_n)y_n+c_ny_{n-1}=0$. Further, from (4.21), we find

$$\Omega(z,\zeta)=-\frac{X_0(z)}{A_0}(\zeta-z)B(X_0(w),(w-\zeta)^{-1})$$

$$-k_0\{X_0(z)\mathcal{B}(P_1(w),(w-\zeta^{-1})-X_1(z)\mathcal{B}(X_0(w),(w-\zeta)^{-1})\}$$
$$=-\frac{X_0(z)}{A_0}(\zeta-z)\mathcal{B}(X_0(w),(w-\zeta)^{-1})-k_0\mathcal{B}(X_0(z),X_1(w)$$
$$-X_1(z)X_0(w),(w-\zeta)^{-1}=\frac{(X_0(w))^2}{A_0}\mathcal{B}(\zeta-z,(w-\zeta),(w-\zeta)^{-1})$$
$$-k_0\mathcal{B}(p_{0,0}(p_{1,0}w+p_{1,1})-(p_{1,0}z+p_{1,1})p_{0,0},(w-\zeta)^{-1})$$

$$= -\frac{(X_0(w))^2}{A_0}\mathcal{B}(\zeta - z, (w-\zeta)^{-1}) + \frac{p_{0,0}}{p_{1,0}A_0}B(p_{0,0}p_{1,0}(w-z), (w-\zeta)^{-1})$$

$$= \frac{(X_0(w))^2}{A_0}\mathcal{B}(w-\zeta, (w-\zeta)^{-1}) = \frac{(X_0(w))^2}{A_0}\mathcal{B}(1,1) = \frac{1}{A_0}\mathcal{B}(X_0(w), X_0(w)) = 1.$$

Thus, (4.22) reduces to

$$\frac{1}{\zeta - z} = \sum_{n=0}^{\nu} \frac{1}{A_n} X_n(z) Y_n(\zeta) + \frac{\Omega_\nu(z,\zeta)}{\zeta - z} \tag{4.24}$$

provided $\zeta \in D$ and $z \neq \zeta$.

It is reasonable to consider the above equality as a formula of Christoffel-Darboux' type for the systems $\{X_n(z)\}_{n=0}^{\infty}$ and $\{Y_n(\zeta)\}_{n=0}^{\infty}$.

4.5 In the cases of the Jacobi, Laguerre and Hermite polynomials and associated functions, the corresponding bilinear forms satisfy all the conditions for existence of a formula of the kind (4.24). Therefore, we can specify it for these systems and as a result we obtain the following representations of the Cauchy kernel:

$$\frac{1}{\zeta - z} = \sum_{n=0}^{\nu} \frac{1}{I_n^{(\alpha,\beta)}} P_n^{(\alpha,\beta)}(z) Q_n^{(\alpha,\beta)}(\zeta) + \frac{\Delta_\nu^{(\alpha,\beta)}(z,\zeta)}{\zeta - z}, \tag{4.25}$$

$$z \neq \zeta \in \mathbb{C} \setminus [-1,1];\ \nu = 0,1,2,\dots,$$

where

$$-\frac{2^{\alpha+\beta+1}(2\nu+\alpha+\beta)\Gamma(\nu+\alpha+1)\Gamma(\nu+\beta+1)}{\Gamma(\nu+2)\Gamma(\nu+\alpha+\beta+2)}\Delta_\nu^{(\alpha,\beta)}(z,\zeta) \tag{4.26}$$

$$= P_\nu^{(\alpha,\beta)}(z) Q_{\nu+1}^{(\alpha,\beta)}(\zeta) - P_{\nu+1}^{(\alpha,\eta)}(z) Q_\nu^{(\alpha,\beta)}(\zeta),$$

$$\frac{1}{\zeta - z} = \sum_{n=0}^{\nu} \frac{1}{I_n^{(\alpha)}} L_n^{(\alpha)}(z) M_n^{(\alpha)}(\zeta) + \frac{\Delta_\nu^{(\alpha)}(z,\zeta)}{\zeta - z}, \tag{4.27}$$

$$z \neq \zeta \in \mathbb{C} \setminus [0,\infty); \nu = 0,1,2,\dots,$$

where

$$\Delta_\nu^{(\alpha)}(z,\zeta) = \frac{\Gamma(\nu+2)}{\Gamma(\nu+\alpha+1)}\{L_\nu^{(\alpha)}(z) M_{\nu+1}^{(\alpha)}(\zeta) - L_{\nu+1}^{(\alpha)}(z) M_\nu^{(\alpha)}(\zeta)\}, \tag{4.28}$$

and

$$\frac{1}{\zeta - z} = \sum_{n=0}^{\nu} \frac{1}{I_n} H_n(z) G_n(\zeta) + \frac{\Delta_\nu(z,\zeta)}{\zeta - z}, \tag{4.29}$$

$$z \neq \zeta \in \mathbb{C} \setminus \mathbb{R};\ \nu = 0, 1, 2, \ldots,$$

where

$$\Delta_\nu(z, \zeta) = -\frac{1}{\sqrt{\pi}\nu! 2^{\nu+1}} \{H_\nu(z) G_{\nu+1}(\zeta) - H_{\nu+1}(z) G_\nu(\zeta)\}. \tag{4.30}$$

5. Relations to hypergeometric and Weber-Hermite functions

5.1 The classical Jacobi, Laguerre and Hermite polynomials appear as particular cases of the hypergeometric and Weber-Hermite functions. In order to clarify this relationship, we begin with the following proposition:

(I.5.1) *Suppose that $A(z) = a_0 + a_1 z$ and $B(z) = b_0 + b_1 z + b_2 z^2$. Then for each $n = 0, 1, 2, \ldots$ the polynomial* (1.2) *is a solution of the differential equation*

$$B(z)u'' + [B'(z) + A(z)]u' - \gamma_n u = 0, \tag{5.1}$$

where

$$\gamma_n = n\{a_0 + (n+1)b_0\},\ n = 0, 1, 2, \ldots. \tag{5.2}$$

Proof. It is sufficient to prove that (5.1) holds in an arbitrary simply connected domain $G \subset \mathbb{C}$ not containing the points z_1, z_2. Since $\deg B \leq 2$, for $z \in G$ the Leibniz rule gives

$$\frac{d^{n+1}}{dz^{n+1}} \{B(z)[B^n(z)w(z)]'\} = B(z)(w(z)W_n(z))'' \tag{5.3}$$

$$+(n+1)B'(z)(w(z)W_n(z))' + \frac{n(n+1)}{2} + B''(z)(w(z)W_n()),\ n = 0, 1, 2, \ldots.$$

On the other hand, the equation (1.1) yields the relation $B(z)[B^n(z)w(z)]'$ $=[nB'(z) + A(z)]B^n(z)w(z)$. The application of Leibniz's rule to it gives

$$\frac{d^{n+1}}{dz^{n+1}} \{B(z)[B^n(z)w(z)]'\} = [nB'(z) + A(z)](w(z)W_n(z))' \tag{5.4}$$

$$+(n+1)[nB''(z) + A'(z)](w(z)W_n(z)),\ n = 0, 1, 2, \ldots.$$

From (5.3) and (5.4) we obtain that

$$B(z)(w(z)W_n(z))'' + [B'(z) - A(z)](w(z)W_n(z))' \tag{5.5}$$

$$-\Big[\frac{n(n+1)}{2} B''(z) + (n+1)A'(z)\Big](w(z)W_n(z)) = 0,\ n = 0, 1, 2, \ldots.$$

Further, using the equation (1.1), we find

$$(w(z)W_n(z))' = w(z)\{P_n'(z) + A(z)(B(z))^{-1}W_n(z)\},$$

$$(w(z)W_n(z))'' = w(z)\{W_n''(z) + 2A(z)(B(z))^{-1}W_n'(z)$$
$$+[A'(z)B(z) - A(z)B'(z) + A^2(z)](B(z))^{-2}W_n(z)\},\ n = 0, 1, 2, \dots$$

Then, substituting the above expression in (5.5) and keeping in mind that $w(z) \neq 0$ in G, we prove the equality

$$B(z)P_n''(z) + [B'(z) + A(z)]P_n'(z) - \gamma_n P_n(z) = 0$$

for $n = 0, 1, 2, \dots$ and $z \in G$. By the identity theorem we conclude that it holds for every $z \in \mathbb{C}$.

The following assertion follows from the general property of polynomials of the kind (1.2) just proved:

(I.5.2): **(a)** *The polynomial* $P_n^{(\alpha,\beta)}(z),\ n = 0, 1, 2, \dots$ *is a solution of the equation*

$$(1 - z^2)w'' + [\beta - \alpha - (\alpha + \beta + 2)z]w' + n(n + \alpha + \beta + 1)w = 0; \tag{5.6}$$

(b) *The polynomial* $L_n^{(\alpha)}(z),\quad n = 0, 1, 2, \dots$ *is a solution of the equation*

$$zw'' + (\alpha + 1 - z)w' + nw = 0; \tag{5.7}$$

(c) *The polynomial* $H_n(z),\ n = 0, 1, 2, \dots$ *is a solution of the equation*

$$w'' - 2zw' + 2nw = 0. \tag{5.8}$$

5.2 We put $z = 1 - 2\zeta$ in (5.6), and obtain the equation

$$\zeta(1 - \zeta)w'' + [\alpha + 1 - (\alpha + \beta + 2)\zeta]w' + n(n + \alpha + \beta + 1)w = 0, \tag{5.9}$$

i.e. a hypergeometric equation with $a = -n, b = n + \alpha + \beta + 1$ and $c = \alpha + 1$ [Appendix, (3.1)]. Suppose that $\alpha+\beta+2$ and $\alpha+1$ are different from $0, -1, -2, \dots$. Then Gauss' hypergeometric function $F(-n, n + \alpha + \beta + 1, \alpha + 1; \zeta)$, which is a polynomial of degree n, and the polynomial $P_n^{(\alpha,\beta)}(1-2\zeta)$, which is of degree n too, are solutions of the equation (5.9). Hence, there exist constants $K_n,\ n = 0, 1, 2, \dots$ such that the relation $P_n^{(\alpha,\beta)}(\zeta) = K_n F(-n, n + \alpha + \beta + 1, \alpha + 1; \zeta)$ holds for each $\zeta \in \mathbb{C}$ and $n = 0, 1, 2, \dots$.

Comparing the coefficients of ζ^n in the last equality and using (2.3) we find that

$$\frac{(-1)^n(n + \alpha + \beta + 1)_n}{n!} = K_n \frac{(-n)_n(n + \alpha + \beta + 1)_n}{n!(\alpha + 1)_n},\ n = 0, 1, 2, \dots,$$

and, hence, $K_n = \binom{n+\alpha}{n}$, $\quad n = 0, 1, 2, \ldots$. Then, returning back to the original variable z, we obtain the representation

$$(5.10) \qquad P_n^{(\alpha,\beta)}(z) = \binom{n+\alpha}{n} F(-n, n+\alpha+\beta+1, \alpha+1; (1-z)/2),$$

$$z \in \mathbb{C};\ n = 0, 1, 2, \ldots,$$

which holds when $\alpha+\beta+2 \neq 0, -1, -2, \ldots$ and $\alpha+1 \neq 0, -1, -2 \ldots$. If the last requirement is not fulfilled but $\beta+1 \neq 0, -1, -2, \ldots$, then likewise we find the representation

$$(5.11) \qquad P_n^{(\alpha,\beta)}(z) = (-1)^n \binom{n+\beta}{n} F(-n, n+\alpha+\beta+1, \beta+1; (1+z)/2),$$

$$z \in \mathbb{C};\ n = 0, 1, 2, \ldots.$$

Remark. If $\alpha+\beta+2 \notin \mathbb{Z}^- \cup \{0\}$, then at least one of the numbers $\alpha+1$ and $\beta+1$ does not belong to $\mathbb{Z}^- \cup \{0\}$, so that either the representation (5.10), or (5.11) is always valid.

5.3 The equation (5.7) is a special case of [Appendix, (3.7)] with $a = -n$ and $c = \alpha+1$. If $\alpha+1 \notin \mathbb{Z}^- \cup \{0\}$, then the function $\Phi(-n, \alpha+1; z)$ is a polynomial of degree n. That is why there exist constants $L_{n,\alpha}$, $n = 0, 1, 2, \ldots$ such that $L_n^{(\alpha)}(z) = L_{n,\alpha}\Phi(-n, \alpha+1; z)$, $z \in \mathbb{C}$, $n = 0, 1, 2, \ldots$. Again, comparing the coefficients of z^n, we find that $L_{n,\alpha} = \binom{n+\alpha}{n}$, $n = 0, 1, 2, \ldots$, i.e.

$$(5.12) \qquad L_n^{(\alpha)}(z) = \binom{n+\alpha}{n} \Phi(-n, \alpha+1; z),\ z \in \mathbb{C};\ n = 0, 1, 2, \ldots.$$

Let us note that the above relations hold when $\alpha+1$ is not equal to $0, -1, -2, \ldots$.

5.4 The equation (5.8) allows us to prove that the polynomial $H_n(\zeta/\sqrt{2})$, $n = 0, 1, 2, \ldots$ is a solution of the equation

$$(5.13) \qquad w'' - \zeta w' + nw = 0.$$

The function $D_n(\zeta)$, $\zeta \in \mathbb{C}$, $n = 0, 1, 2, \ldots$ [Appendix, (4.2)] is a solution of the equation $u'' + (n+1/2-\zeta^2/4)u = 0$. If we substitute $u = \exp(-\zeta^2/4)w$, then it turns out that the function $w(\zeta)$ is a solution of (5.13). In particular, the function $\exp(\zeta^2/4)D_n(\zeta)$, which is a polynomial of degree n, is a solution of (5.13). Hence, there exist constants M_n, $n = 0, 1, 2, \ldots$ such that

$$(5.14) \qquad H_n(\zeta/\sqrt{2}) = M_n \exp(\zeta^2/4) D_n(\zeta), \zeta \in \mathbb{C};\ n = 0, 1, 2, \ldots.$$

Suppose that $n = 2m,\ m = 0,1,2,\dots$ and put $\zeta = 0$ in (5.14). Then from [Appendix, (4.2)] and (2.12) it follows that

$$\frac{(-1)^m(2m)!}{m!} = \frac{2^m\Gamma(1/2)M_{2m}}{\Gamma(1/2-m)},\ m = 0,1,2,\dots. \tag{5.15}$$

From the multiplication formula [Appendix, (1.10)]

$$\Gamma(1/2-m)\Gamma(1/2+m) = \frac{\pi}{\sin((m+1/2)\pi)} = (-1)^m\pi,\ m = 0,1,2,\dots,$$

we easily obtain that

$$\Gamma(1/2-m) = \frac{(-1)^m\pi m!2^{2m}}{(2m)!\Gamma(1/2)},\ m = 0,1,2,\dots.$$

Now it is clear that (5.15) means that $M_{2m} = 2^m,\ m = 0,1,2,\dots$.

The representations (2.12) and [Appendix, (4.2),(3.8)] allows by differentiation of (5.14) followed by setting $\zeta = 0$ to find that $M_{2m+1} = 2^{m+1/2},\ m = 0,1,2,\dots$. Hence, $M_n = 2^{n/2},\ n = 0,1,2,\dots$, and then (5.14) yields

$$H_n(z) = 2^{n/2}\exp(z^2/2)D_n(z\sqrt{2}),\ z \in \mathbb{C};\ n = 0,1,2,\dots. \tag{5.16}$$

5.5 Now we are going to show that the Jacobi, Laguerre and Hermite associated functions also can be expressed in terms of hypergeometric and Weber-Hermite functions.

(I.5.3) *If $\alpha+1, \beta+1$ and $\alpha+\beta+2$ do not belong to $\mathbb{Z}^- \cup \{0\}$, then the representation*

$$\frac{\Gamma(2n+\alpha+\beta+2)(z-1)^{n+1}Q_n^{(\alpha,\beta)}(z)}{2(n+\alpha+\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \tag{5.17}$$

$$= F(n+\alpha+1, n+1, 2n+\alpha+\beta+1; 2/(1-z)),\ n = 0,1,2,\dots$$

holds in the region $\mathbb{C}\setminus[-1,1]$.

Proof. Suppose that $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\beta > 0$. Then from the integral representation (4.4) it follows that for $z \in \mathbb{C}\setminus[-1,1]$ and $n = 0,1,2,\dots$,

$$2^n(z-1)^{n+1}Q_n^{(\alpha,\beta)}(z) = \int_{-1}^{1}\frac{(1-t)^{n+\alpha}(1+t)^{n+\beta}}{(1+(1-t)/(z-1))^{n+1}}\,dt. \tag{5.18}$$

Since $|(1-t)/(z-1)| \le 2|z-1|^{-1} < 1$ when $|z-1| > 2$ and $t \in [-1,1]$, the series

$$\sum_{k=0}^{\infty}(-1)^k\binom{n+k}{n}\Big(\frac{1-t}{z-1}\Big)^k$$

is uniformly convergent with respect to $t \in [-1,1]$. Since $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\beta > 0$, it is clear that the series which is obtained by termwise multiplication of the above series by the function $(1-t)^{n+\alpha}(1+t)^{n+\beta}$ is also uniformly convergent on the segment $[-1,1]$. Since the sum of the above series is $(1+(1-t)(z-1)^{-1})^{-n-1}$, (5.18) yields that

$$2^n (z-1)^{n+1} Q_n^{(\alpha,\beta)}(z) \tag{5.19}$$
$$= \sum_{k=0}^{\infty} (-1)^k \binom{n+k}{n} (z-1)^{-k} \int_{-1}^{1} (1-t)^{n+k+\alpha}(1+t)^{n+\beta}\,dt.$$

But

$$\int_{-1}^{1} (1-t)^{n+k+\alpha}(1+t)^{n+\beta}\,dt$$
$$= \frac{2^{2n+k+\alpha+\beta+1}\Gamma(n+k+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+k+\alpha+\beta+2)},\ n,k = 0,1,2,\dots,$$

and then (5.19) and [Appendix, (3.6)] lead to the representation (5.17).

So far the validity of (5.17) is established provided $|z-1| > 2, \operatorname{Re}\alpha > 0$ and $\operatorname{Re}\beta > 0$. But if n is fixed, then both sides of (5.17) are holomorphic function in the region $\overline{\mathbb{C}} \setminus [-1,1]$, and, by the identity theorem, they coincide in this region. This is true when $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\beta > 0$. If $n = 0,1,2,\dots$ and $z \in \overline{\mathbb{C}} \setminus [-1,1]$ are fixed, then both sides of (5.17) are holomorphic functions of the complex variables α and β provided $\alpha+1 \neq 0,-1,-2,\dots,\beta+1 \neq 0,-1,-2,\dots$ and $\alpha+\beta+2 \neq 0,-1,-2,\dots$. Again the identity theorem yields the validity of (5.17) if $z \in \overline{\mathbb{C}} \setminus [-1,1]$ and none of the complex numbers $\alpha+1, \beta+1$ and $\alpha+\beta+2$ is equal to $0,-1,-2,\dots$.

Remark. As a corollary of (4.2) and (5.17) we have that

$$\varphi(\alpha,\beta;z) = -\frac{2\alpha+\beta+2\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)(1-z)} F(\alpha+1,1,\alpha+\beta+2;2/(1-z)) \tag{5.20}$$

for $z \in \overline{\mathbb{C}} \setminus [-1,1]$ provided none of the complex numbers $\alpha+1, \beta+1$ and $\alpha+\beta+2$ is equal to $0,-1,-2,\dots$.

5.6 Define a and c as $a = n+\alpha+1$ and $c = \alpha+1$. Then from (4.13) and [Appendix, (3.19)] we obtain

$$M_n^{(\alpha)}(z) = -\Gamma(n+\alpha+1)(-z)^{\alpha}\Psi(n+\alpha+1,\alpha+1;z), \tag{5.21}$$

provided $\operatorname{Re}\alpha > -1$ and $z \in \mathbb{C} \setminus [0,\infty)$.

5.7 Now, we emphasize that in fact by means of the equalities (4.16) there are defined the following two sequences of holomorphic functions:

$$G_n^{+}(z) = -\int_{-\infty}^{\infty} \frac{\exp(-t^2)H_n(t)}{t-z}\,dt,\ \operatorname{Im} z > 0,\ n = 0,1,2,\dots, \tag{5.22}$$

and

$$G_n^-(z) = -\int_{-\infty}^{\infty} \frac{\exp(-t^2)H_n(t)}{t-z}\,dt,\ \operatorname{Im} z < 0,\ n = 0,1,2,\dots. \tag{5.23}$$

It is easy to prove that each of the functions $G_n^+(z)$, $n = 0,1,2,\dots$ and $G_n^-(z)$, $n = 0,1,2,\dots$ is analytically continuable in the whole complex plane. This can be done by means of shifts parallel to the real axis and applying the Cauchy integral theorem. Now, we are going to follow another idea in order to find appropriate relations between the functions (5.22), (5.23) and the Weber-Hermite functions.

From the equalities (2.8) and (2.12) we obtain

$$H_{2n}(z) = (-1)^n n! 2^{2n} L_n^{(-1/2)}(z^2),\ n = 0,1,2,\dots, \tag{5.24}$$

and

$$H_{2n+1}(z) = (-1)^n n! 2^{2n+1} z L_n^{(1/2)}(z^2),\ n = 0,1,2,\dots. \tag{5.25}$$

Since H_{2n} is an even and H_{2n+1} is an odd polynomial, we easily find that if $\operatorname{Im} z > 0$ and, hence, $z^2 \in \mathbb{C} \setminus [0,\infty)$, then

$$G_{2n}^+(z) = (-1)^n n! 2^{2n} z M_n^{(-1/2)}(z^2),\ n = 0,1,2,\dots \tag{5.26}$$

and

$$G_{2n+1}^+(z)(-1)^n n! 2^{2n+1} M_n^{(1/2)}(z^2),\ n = 0,1,2,\dots. \tag{5.27}$$

Further, from (5.21) and [Appendix, (4.2)] we have

$$M_n^{(1/2)}(z^2) = -\Gamma(n+3/2)2^{n+1}\exp(-z^2/2)D_{-2n-2}(iz\sqrt{2}),\ n = 0,1,2,\dots.$$

Using the relation $\Psi(a,c;z) = z^{1-c}\Psi(a-c+1, 2-c; z)$ [Bateman, H., A.Erdélyi, 1, 6.5.,(6)], we get

$$zM_n^{(-1/2)}(z^2) = i\Gamma(n+1/2)2^{n+1/2}\exp(-z^2/2)D_{-2n-1}(iz\sqrt{2}),\ n = 0,1,2,\dots.$$

Then the equalities (5.26) and (5.27) yield

$$G_{2n}^+(z) = i(-1)^n n!\Gamma(n+1/2)2^{3n+1/2}D_{-2n-1}(iz\sqrt{2}),\ n = 0,1,2,\dots$$

and

$$G_{2n+1}^+(z) = -(-1)^n n!\Gamma(n+3/2)2^{3n+3/2}D_{-2n-2}(iz\sqrt{2}),\ n = 0,1,2,\dots.$$

The above relations could be "unified" by means of the well-known formula $\Gamma(2s) = 2^{2s-1}\pi^{-1/2}\Gamma(s)\Gamma(s+1/2)$ just by setting $s = n+1/2$. Thus, we obtain

$$G_n^+(z) = (2\pi)^{1/2}\exp(-z^2)i^{n+1}n!2^{n/2}D_{-n-1}(iz\sqrt{)}2,\ n = 0,1,2,\dots. \tag{5.28}$$

Since $G_n^-(z) = \overline{G_n^+(\overline{z})}$, $\quad n = 0,1,2,\dots$, as well as $D_\nu(\zeta) = \overline{D_\nu(\overline{\zeta})}$, provided that ν is real, we can also write the identity $(n = 0,1,2,\dots)$

$$G_n^-(z) = (2\pi)^{1/2}\exp(-z^2/2)(-i)^{n+1}n!2^{n/2}D_{-n-1}(-iz\sqrt{2}). \tag{5.29}$$

Exercises

1. Prove the relations $P_n^{(\alpha,\beta)}(z) = (-1)^n P_n^{(\beta,\alpha)}(-z),\ n = 0,1,2,\dots.$

2. Prove the validity of the equalities

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n},\ P_n^{(\alpha,\beta)}(-1) = (-1)^n\binom{n+\beta}{n},\ n = 0,1,2,\dots,$$

as well as their equivalence.

3. Prove the following relations and their equivalence:

$$(\mathbf{a})\ (2n+\alpha+\beta+2)(1-z)P_n^{(\alpha+1,\beta)}(z)$$

$$= 2\{(n+\alpha+1)P_n^{(\alpha,\beta)}(z) - (n+1)P_{n+1}^{(\alpha,\beta)}(z)\},\ n = 0,1,2,\dots;$$

$$(\mathbf{b})\ 2(n+\alpha+\beta+2)(1+z)P_n^{(\alpha,\beta+1)}(z)$$

$$= 2\{(n+\beta+1)P_n^{(\alpha,\beta)}(z) + n+1P_{n+1}^{(\alpha,\beta)}(z)\},\ n = 0,1,2,\dots$$

4. Prove: $\{P_n^{(\alpha,\beta)}(z)\}' = (1/2)(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(z),\ n = 1,2,3,\dots.$

5. Prove:

$$(\mathbf{a})\ L_n^{(\alpha)}(z) = L_n^{(\alpha+1)}(z) - L_{n-1}^{(\alpha+1)}(z),\ n = 1,2,3,\dots;$$

$$(\mathbf{b})\ zL_n^{(\alpha)}(z) = (n+\alpha+1)L_n^{(\alpha)}(z) - (n+1)L_n^{(\alpha)}(z);$$

$$(\mathbf{c})\ L_n^{(\alpha)}(z) = \frac{(-1)^n}{n!(-z)^\alpha\exp(-z)}\{(-z)^{n+\alpha}\exp(-z)\}^{(n)},$$

$$z \in \mathbb{C}\setminus[0,\infty),\ n = 0,1,2,\dots.$$

6. Prove: $\{L_n^{(\alpha)}(z)\}' = -L_{n-1}^{(\alpha+1)}(z),\ n = 1,2,3,\dots.$

7. Prove: $L_n^{(-k)}(z) = (-z)^k\frac{(n-k)!}{n!}L_{n-k}^{(k)}(z),\ n = k,k+1,k+2,\dots.$

8. Prove:

$$H_{2n}(0) = (-1)^n\frac{(2n)!}{n!},\ H_{2n+1}'(0) = (-1)^n\frac{(2n+2)!}{(n+1)!},\ n = 0,1,2,\dots.$$

9. Prove: $\sum_{k=0}^{n} \binom{n}{k} H_k(z)H_{n-k}(w) = 2^{n/2}H_n((z+w)/2),\ n=0,1,2,\dots.$

10.Prove: $H_n'(z) = 2nH_{n-1}(z),\ n=1,2,3,\dots.$

11. Prove:

$$(\mathbf{a}) \int_{-\infty}^{\infty} \exp(-t^2)H_{2n}(zt)\,dt = \sqrt{\pi}\frac{(2n)!}{n!}(z^2-1)^n,\ n=0,1,2,\dots;$$

$$(\mathbf{b}) \int_{-\infty}^{\infty} \exp(-t^2)tH_{2n+1}(zt)\,dt = \sqrt{\pi}\frac{(2n+1)!}{n!}z(z^2-1)^n,\ n=0,1,2,\dots.$$

12.Prove that for each $n=0,1,2,\dots$ the entire function $\exp(-z^2/2)H_n(z)$ is a solution of the differential equation $w''+(2n+1-z^2)w=0$.

13. Suppose that $A_j,\ j=0,1,2$ and F are complex-valued functions holomorphic in a simply connected region $G\subset\mathbb{C}$ and that $A_0(z)\neq 0$ for $z\in G$. Let u_1 and u_2 be linearly independent solutions of the differential equation $A_0(z)w''+A_1(z)w'+A_2(z)w=0$ in the region G. Show that for arbitrary complex constants $C_j,\ j=1,2$ and any point $z_0\in G$:

(a) The function

$$u(z) = C_1u_1(z)+C_2u_2(z)+\int_{z_0}^{z} \frac{u_1(z)u_2(\zeta)-u_1(\zeta)u_2(z)}{u_1'(\zeta)u_2(\zeta)-u_1(\zeta)u_2'(\zeta)}\cdot\frac{F(\zeta)}{A_0(\zeta)}\,d\zeta,\ z\in G$$

is a solution of the equation $A_0(z)w''+A_1(z)w'+A_2(z)w=F(z)$ in the domain G;

$$(\mathbf{b})\ W(u_1,u_2;z) = u_1'(z)u_2(z)-u_1(z)u_2'(z) = W(u_1,u_2;z_0)\exp\left\{-\int_{z_0}^{z}\frac{A_1(\zeta)}{A_0(\zeta)}\,d\zeta\right\}.$$

Remark. The representation **(a)** and **(b)** are known as Liouville's formulas.

14. Prove that for each $n=0,1,2,\dots$ and $z\in\mathbb{C}$ the equality

$$\exp(-z^2/2)H_n(z) = \lambda_n\cos\{(2n+1)^{1/2}z-n\pi/2\}$$

$$+(2n+1)^{-1/2}\int_0^z \sin\{(2n+1)^{1/2}(z-\zeta)\}\zeta^2\exp(-\zeta^2/2)H_n(\zeta)\,d\zeta$$

holds with $\lambda_{2n}=\Gamma(2n+1)/\Gamma(n+1)$ and $\lambda_{2n+1}=(2n+1)^{-1/2}\Gamma(2n+3)/\Gamma(n+2)$.

Remark. It means that the entire function $\exp(-z^2/2)H_n(z)$ is a solution of a Voltera's integral equation of the second kind.

15. The polynomials $\{P_n^{(\lambda)}(z)\}_{n=0}^{\infty}$, defined by

$$P_n^{(\lambda)}(z) = \frac{\Gamma(\lambda+1/2)\Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma(n+\lambda+1/2)}P_n^{(\lambda-1/2,\lambda-1/2)}(z),$$

$$2\lambda \neq 0, -1, -2, \dots,\ n = 1, 2, 3, \dots,$$

$$P_n^{(0)}(z) = \lim_{\lambda \to \infty} \lambda^{-1} P_n^{(\lambda)}(z),\ n = 1, 2, 3, \dots,$$

and $P_0^{(\lambda)}(z) \equiv 1$, are called ultraspherical (sometimes Gegenbauer) polynomials. Show that:

(a) $\deg P_n^{(\lambda)} = n,\ n = 0, 1, 2, \dots;$

(b) if $\operatorname{Re}\lambda > -1/2$ and $\lambda \neq 0$, then

$$\int_{-1}^{1} (1-t^2)^{\lambda-1/2} P_m^{(\lambda)}(t) P_n^{(\lambda)}(t)\, dt = \frac{2^{1-2\lambda}\pi\Gamma(n+2\lambda)}{n!(n+\lambda)\{\Gamma(\lambda)\}^2}\delta_{mn},\ m, n = 0, 1, 2, \dots;$$

(**c**) $$P_n^{(\lambda)}(z) = \frac{(-1)^n\Gamma(\lambda+1/2)\Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma(n+\lambda+1/2)2^n n!(1-z^2)^{\lambda-1/2}}\frac{d^n}{dz^n}\{(1-z^2)^{n+\lambda-1/2}\},$$

$$z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\},\ n = 0, 1, 2, \dots;$$

(**d**) $$P_n^{(\lambda)}(z) = \sum_{k=0} [n/2] \frac{(-1)^k\Gamma(n-k+\lambda)}{\Gamma(\lambda)k!(n-2k)!}(2z)^{n-2k},$$

$$\lambda \neq 0, -1, -2, \dots,\ n = 0, 1, 2, \dots;$$

(e) $P_n^{(\lambda)}(z)$ is a solution of the differential equation $(1-z^2)w'' - (2\lambda)zw'$ $+n(n+2\lambda)w = 0;$

(**f**) $$P_n^{(\lambda)}(z) = F(-n, n+2\lambda, \lambda+1/2; (1-z)/2),\ n = 0, 1, 2, \dots.$$

16. The polynomials $\{P_n(z)\}_{n=0}^{\infty}$, defined by $P_n(z) = P_n^{(1/2)}(z) = P_n^{(0,0)}(z)$, $n = 0, 1, 2, \dots$, are called Legendre polynomials. Show that:

(**a**) $$\int_{-1}^{1} P_m(t)P_n(t)\, dt = \frac{2}{2n+1}\delta_{mn}, m, n = 0, 1, 2, \dots;$$

(**b**) $$P_n(z) = \frac{(-1)^n}{n!2^n}\frac{d^n}{dz^n}\{(1-z^2)^n\},\ z \in \mathbb{C},\ n = 0, 1, 2, \dots;$$

(**c**) $$P_n(z) = 2^{-n}\sum_{k=0}^{[n/2]}(-1)^k \binom{n}{k}\binom{2n-2k}{n} z^{n-2k},\ n = 0, 1, 2, \dots;$$

(d) $P_n(z)$ satisfies the differential equation $(1-z^2)w'' - 2zw' + n(n+1)w = 0;$

(**e**) $$P_n(z) = F(-n, n+1, 1; (1-z)/2),\ n = 0, 1, 2, \dots.$$

17. The polynomials $\{T_n(z)\}_{n=0}^{\infty}$, defined by the equalities $T_n(z)$

$= (n/2)P_n^{(0)}(z) = g_n^{-1}P_n^{(-1/2,-1/2)}(z),\ n = 0, 1, 2, \ldots$,where

$$g_n = 2^{-2n}\binom{2n}{n} = \frac{\Gamma(2n+1)}{2^{2n}\{\Gamma(n+1)\}^2},\ n = 0, 1, 2, \ldots,$$

are called Chebyshev polynomials of the first kind. Show that:

$$(\mathbf{a})\ \int_{-1}^{1}(1-t^2)^{-1/2}T_m(t)T_n(t)\,dt = \tau_n\delta_{mn}, m, n = 0, 1, 2, \ldots,$$

where $\tau_0 = \pi$ and $\tau_n = \pi/2,\ n = 1, 2, 3 \ldots$.

$$(\mathbf{b})\ T_n(z) = \frac{(-1)^n\Gamma(1/2)(1-z^2)^{1/2}}{2^n\Gamma(n+1/2)}\frac{d^n}{dz^n}\{(1-z^2)^{n-1/2}\},$$

$$z \in \mathbb{C}\setminus\{(-\infty,-1]\cup[1,\infty)\},\ n = 0, 1, 2, \ldots;$$

$$(\mathbf{c})\ T_n(z) = \frac{n}{2}\sum_{k=0}^{[n/2]}\frac{(-1)^k(n-k-1)!}{k!(n-2k)!}(2z)^{n-2k},\ n = 1, 2, 3, \ldots;$$

(d) $T_n(z)$ is a solution of the differential equation $(1-z^2)w'' - zw' + n^2w = 0$;

$$(\mathbf{e})\ T_n(z) = F(-n, n, 1/2; (1-z)/2),\ n = 0, 1, 2, \ldots.$$

18. The polynomials $\{U_n(z)\}_{n=0}^{\infty}$, defined by the equalities $U_n(z) = P_n^{(1)}(z)$ $= (2g_n)^{-1}P_n^{(1/2,1/2)}(z),\ n = 0, 1, 2, \ldots$, are called Chebyshev polynomials of the second kind. Show that:

$$(\mathbf{a})\ \int_{-1}^{1}(1-t^2)^{1/2}U_m(t)U_n(t)\,dt = (\pi/2)\delta_{mn},\ m, n = 0, 1, 2, \ldots;$$

$$(\mathbf{b})\ U_n(z) = \frac{(-1)^n(n+1)\sqrt{\pi}}{(2n+1)2^{n+1}\Gamma(n+1/2)(1-z^2)^{1/2}}\frac{d^n}{dz^n}\{(1-z^2)^{n+1/2}\},$$

$$z \in \mathbb{C}\setminus\{(-\infty,-1]\cup[1,\infty)\},\ n = 0, 1, 2, \ldots;$$

$$(\mathbf{c})\ U_n(z) = \sum_{k=0}^{[n/2]}\frac{(-1)^k(n-k)!}{k!(n-2k)!}(2z)^{n-2k},\ n = 0, 1, 2, \ldots;$$

(d) $U_n(z)$ satisfies the differential equation $(1-z^2)w'' - 3zw' + n(n+2)w = 0$;

$$(\mathbf{e})\ U_n(z) = F(-n, n+1, 3/2; (1-z)/2),\ n = 0, 1, 2, \ldots.$$

19. Suppose that γ is a rectifiable Jordan arc "connecting" the points -1 and 1 in the region $G = \mathbb{C}\setminus\{(-\infty,-1]\cup[1,\infty)\}$, i.e., with exception of these points, γ is contained in G. Prove that

$$\int_{\gamma}(1-z)^{\alpha}(1+z)^{\beta}P_m^{(\alpha,\beta)}(z)P_n^{(\alpha,\beta)}(z)\,dz = I_n^{(\alpha,\beta)}\delta_{mn},\ m, n = 0, 1, 2, \ldots.$$

20. Let l_θ, $-\pi/2 < \theta < \pi/2$, be the ray $z = t\exp i\theta$, $0 \le t < \infty$. Prove that if $\operatorname{Re}\alpha > -1$, then

$$\int_{l_\theta} z^\alpha \exp(-z) L_m^{(\alpha)}(z) L_n^{(\alpha)}(z) = I_n^{(\alpha)} \delta_{mn},\ m, n = 0, 1, 2, \dots.$$

21. Show that the system of ultraspherical polynomials $\{P_n^{(\lambda)}(z)\}_{n=0}^\infty$ is a solution of the recurrence equation $(n+1)y_{n+1} - 2(n+\lambda)zy_n + (n+2\lambda-1)y_{n-1} = 0$.

22. Show that the system of Legendre polynomials $\{P_n(z)\}_{n=0}^\infty$ is a solution of the recurrence equation $(n+1)y_{n+1} - 2(n+1)zy_n + ny_{n-1} = 0$.

23. Show that each of the systems of Chebyshev polynomials $\{T_n(z)\}_{n=0}^\infty$ and $\{U_n(z)\}_{n=0}^\infty$ is a solution of the recurrence equation $y_{n+1} - 2zy_n + y_{n-1} = 0$.

24. Prove that $Q_n^{(\alpha,\beta)}(-z) = (-1)^{n+1} Q_n^{(\beta,\alpha)}(z), z \in \mathbb{C} \setminus [-1,1], n = 0, 1, 2, \dots$.

25. Prove that if $\alpha \in \mathbb{C} \setminus \mathbb{Z}^-$, then for each $z \in \mathbb{C} \setminus [0,\infty)$ and $n = 1, 2, 3, \dots$, $(M_{n-1}^{(\alpha+1)}(z))' = nM_n^{(\alpha)}(z)$.

26. Prove that the representation (4.10) holds when $\operatorname{Re}(n+\alpha) > -1$.

27. Prove the validity of the following identities:

$$(\mathbf{a})\ \Delta_\nu^{(\alpha,\beta)}(z,z) \equiv 1,\ z \in \mathbb{C} \setminus [-1,1],\ \nu = 0, 1, 2, \dots;$$

$$(\mathbf{b})\ \Delta_\nu^{(\alpha)}(z,z) \equiv 1,\ z \in \mathbb{C} \setminus [0,\infty),\ \nu = 0, 1, 2, \dots;$$

$$(\mathbf{c})\ \Delta_\nu(z,z) \equiv 1,\ z \in \mathbb{C} \setminus \mathbb{R}, \nu = 0, 1, 2, \dots.$$

28. Suppose that $\tau \in \mathbb{R}$ and denote by $l(\tau)$ the line with parametric equation $\zeta = t + i\tau, -\infty < t < \infty$. Prove that

$$G_n^+(z) = -\int_{l(\tau)} \frac{\exp(-\zeta^2) H_n(\zeta)}{\zeta - z}\, d\zeta,\ \operatorname{Im} z > \tau,\ n = 0, 1, 2, \dots,$$

and

$$G_n^-(z) = -\int_{l(\tau)} \frac{\exp(-\zeta^2) H_n(\zeta)}{\zeta - z}\, d\zeta,\ \operatorname{Im} z < \tau,\ n = 0, 1, 2, \dots.$$

29. Prove that for $z \in \mathbb{C}$ and $n = 0, 1, 2, \dots$,

$$G_n^-(z) - G_n^+(z) = 2\pi i \exp(-z^2) H_n(z),\ n = 0, 1, 2, \dots.$$

30. Prove that for $z \in \mathbb{C}$ and $n = 0, 1, 2, \dots$,

$$\exp(-z^2) H_n(z) = \frac{2^{n+1}}{\sqrt{\pi}} \int_0^\infty \exp(-t^2) t^n \cos(2zt - n\pi/2)\, dt.$$

Comments and references

We define the classical polynomials of Jacobi $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$, Laguerre $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ and Hermite $\{H_n(z)\}_{n=0}^{\infty}$ by means of Rodrigues' type formulas as in the book of F.TRICOMI [1]. Thus, we avoid the restrictions on the parameters α and β in defining Jacobi and Laguerre polynomials as orthogonal systems on intervals of the real axis with respect to suitable weight functions. As it is shown by Tricomi this is a direct way to conclude that a polynomial defined by equality (1.2) is a solution of the linear differential equation of second order (5.1). The last property leads further to the well-known representations of the classical orthogonal polynomials in terms of hypergeometric functions.

The orthogonality of Jacobi as well as of Laguerre polynomials is studied by many authors. Among them are N. OBRECHKOFF [1], I. BAIČEV [1], P. RUSEV [1], A.M. KRALL [1] and recently M.E.H. ISMAIL, D.R. MASSON AND M. RAHMAN [1]. In the paper of S.J.L. VAN EIJNDHOVEN and J.L.H. MEYERS [1] it is proved that the system of functions $\{H_n(x+iy)\}_{n=0}^{\infty}$, $x, y \in \mathbb{R}$ is orthogonal on $\mathbb{R}^2$ with respect to a family of weight-functions depending on a real parameter.

The idea to define orthogonality of algebraic polynomials with respect to a bilinear form with the property indicated in **(I.3.9)**, which is not new (see e.g. A.J. DURAN [1]), seems to be very useful. In any case it provides the well-known general property that a system of orthogonal polynomials is a solution of a linear recurrence equation of second order. The acceptance of this approach, specified to the Jacobi, Laguerre and Hermite polynomials, enables us to compute all the constants involved in the corresponding recurrence relations even in the case when the parameters α and β are arbitrary complex numbers.

By convention, the systems of Jacobi, Laguerre and Hermite associate functions are holomorphic solutions of the recurrence equations for the systems of Jacobi, Laguerre and Hermite polynomials. It is worth saying that in this book we do not need the Jacobi, Laguerre and Hermite functions of second kind defined as second solutions of the corresponding differential equations.

Let us point out that, e.g., the Jacobi functions of second kind are (multivalued) analytic functions with branch points at -1 and 1 and that, in general, they have no single-valued branches in the region $\mathbb{C} \setminus [-1, 1]$. As analytic functions with a branch point at the origin, the Laguerre functions of second kind share the same property.

Chapter II

INTEGRAL REPRESENTATIONS AND GENERATING FUNCTIONS

1. Integral representations and generating functions for Jacobi polynomials

1.1 We define $l(1;z) : \zeta = 1 + t(1-z)$ and $l(-1;z) : \zeta = -1 - t(1+z)$ for $0 \le t < \infty$ and $z \ne \pm 1$ as well as

$$\left(\frac{1-\zeta}{1-z}\right)^{\alpha} = \exp\left\{\alpha \log \frac{1-\zeta}{1-z}\right\}, \tag{1.1}$$

for $\zeta \in S(1;z) := \mathbb{C} \setminus l(1;z)$, and

$$\left(\frac{1+\zeta}{1+z}\right)^{\beta} = \exp\left\{\beta \log \frac{1+\zeta}{1+z}\right\}, \tag{1.2}$$

for $\zeta \in S(-1;z) := \mathbb{C} \setminus l(-1;z)$, and arbitrary complex numbers α and β.

It is clear that $S(1;x) = \mathbb{C} \setminus [1,\infty)$ and $S(-1;x) = \mathbb{C} \setminus (-\infty, 1]$ for $x \in (-1,1)$. Due to (1.1) and (1.2),

$$\left(\frac{1-\zeta}{1-x}\right)^{\alpha} = \frac{(1-\zeta)^{\alpha}}{(1-x)^{\alpha}} \tag{1.3}$$

and

$$\left(\frac{1+\zeta}{1+x}\right)^{\beta} = \frac{(1+\zeta)^{\beta}}{(1+x)^{\beta}} \tag{1.4}$$

for $\zeta \in S(1;x) \cap S(-1;x) = \mathbb{C} \setminus \{(-\infty,-1] \cup [1,\infty)\}$ and $x \in (-1,1)$.

(II.1.1) *Let* γ *be a rectifiable Jordan curve such that* $\gamma \cup \operatorname{int}\gamma \subset \mathbb{C} \setminus \{l(1;z) \cup l(-1;z)\}$, *where* $z \ne \pm 1$, *and* $\operatorname{ind}(\gamma; z) = 1$. *Then*

$$P_n^{(\alpha,\beta)}(z) = \frac{1}{2\pi i} \int_{\gamma} \left\{\frac{\zeta^2 - 1}{2(\zeta - z)}\right\}^n \left(\frac{1-\zeta}{1-z}\right)^{\alpha} \left(\frac{1+\zeta}{1+z}\right)^{\beta} \frac{d\zeta}{\zeta - z}. \tag{1.5}$$

Proof. Let $a \ne \pm 1$, $r(a) := \min\{2^{-1}|1-a|, 2^{-1}|1+a|\}$ and

$$B(a) = \mathbb{C} \setminus \bigcup_{|z-a| \le r(a)} \{l(1:z) \cup l(-1;z)\}.$$

Denote the right-hand side of (1.5) by $\tilde{P}_n^{(\alpha,\beta)}(z)$. Then, for $z \in U(a; r(a))$

$:= \{\zeta : |\zeta - a| < r(a)\}$ and $n = 0, 1, 2, \ldots,$

$$\tilde{P}_n^{(\alpha,\beta)}(z) = \frac{1}{2\pi i}\int_{C(a;r(a))}\left\{\frac{\zeta^2-1}{2(\zeta-z)}\right\}^n\left(\frac{1-\zeta}{1-z}\right)^\alpha\left(\frac{1+\zeta}{1+z}\right)^\beta\frac{d\zeta}{\zeta-z},$$

where $C(a;r(a)) := \{\zeta : |\zeta - a| = r(a)\}$.

If $\zeta \in B(a)$, then the left-hand sides of (1.1) and (1.2) are holomorphic functions of z in the disk $U(a;r(a))$. Hence, $\tilde{P}_n^{(\alpha,\beta)}(z)$ is a holomorphic function of z in $U(a;r(a))$, and since a is an arbitrary point of the region $\mathbb{C}\setminus\{-1,1\}$, $\tilde{P}_n^{(\alpha,\beta)}(z)$ is a holomorphic function of z in this region.

If $z = x \in (-1,1)$, then due to [I, (2.1)] and the equalities (1.3) and (1.4), $\tilde{P}_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)$. Referring to the identity theorem, we conclude that $\tilde{P}_n^{(\alpha,\beta)}(z) = P_n^{(\alpha,\beta)}(z)$ for $z \in \mathbb{C}\setminus\{-1,1\}$ and $n = 0, 1, 2, \ldots$.

1.2 There exists unique complex function h, holomorphic in the region $H = \mathbb{C}\setminus[-1,1]$, such that $h^2(z) = z^2 - 1$ when $z \in H$ and $h(x) > 0$ when $x > 1$. Usually, the value of this function at any point $z \in H$ is denoted by $\sqrt{z^2-1}$. The function ω defined in H as

$$\omega(z) = z + \sqrt{z^2-1},\ z \in \mathbb{C}\setminus[-1,1] \tag{1.6}$$

is also holomorphic in H. Moreover, $\omega(z) \neq 0$ and $(\omega(z) + (\omega(z))^{-1})/2 = z$ when $z \in H$. Hence, $\omega(z)$ is as an inverse function of the Zhukovskii function $z = (\omega + \omega^{-1})/2$. As it is well-known, the last one is univalent in the domain $D = \{\omega : |\omega| > 1\}$ and maps it onto H. Hence, the function ω maps H onto D. Since $\lim_{z\to\infty}\omega(z) = \infty$, ω is a meromorphic function in the region $\overline{\mathbb{C}}\setminus[-1,1]$ with a (simple) pole at the point of infinity.

If $z \in H$, then define the function $p(z,w)$ in the disk $U(0;|\omega(z)|^{-1})$ by the requirements $p^2(z,w) = 1 - 2zw + w^2$ and $p(z,0) = 1$. We denote this function by $\sqrt{1-2zw+w^2}$. Its existence follows from the fact that the disk $U(0;|\omega(z)|^{-1})$ is a simply connected region and $1 - 2zw + w^2 \neq 0$ whenever w is in this disk, and $z \in H$. Indeed, the equality $1 - 2zw + w^2 = 0$ implies $w = \omega(z)$ or $w = (\omega(z))^{-1}$ which is impossible.

Notice that the function $1 + p(z,w)$ does not vanish in the disk $U(0;|\omega(z)|^{-1})$ when $z \in H$. Indeed, the equality $p(z_0,w_0) = -1$ implies $w_0(w_0 - 2z_0) = 0$ which contradicts to $p(z_0,0) = p(z_0,2z_0) = 1$. Hence, the function

$$\zeta(w) = \frac{2z-w}{1+p(z,w)} \tag{1.7}$$

is holomorphic in the disk $U(0;|\omega(z)|^{-1})$ for $z \in H$.

If $w \neq 0$, then from (1.7) it follows that $\zeta(w) = (1 - p(z,w))w^{-1}$ and, hence, it hold the equalities

$$(1 - w + p(z,w))(1 - \zeta(w)) = 2(1 - z), \tag{1.8}$$

and

$$(1 + w + p(z,w))(1 + \zeta(w)) = 2(1 + z) \tag{1.9}$$

A direct verification shows that these equalities are still valid even for $w = 0$. Moreover, as their implications we obtain that $1 - w + p(z,w) \neq 0$ and $1 + w + p(z,w) \neq 0$ for $z \in H$ and $w \in U(0; |\omega(z)|^{-1})$.

Introduce the function $P^{(\alpha,\beta)}(z,w)$ for $z \in H$ and $w \in U(0; |\omega(z)|^{-1}) \setminus \{0\}$ by

$$P^{(\alpha,\beta)}(z,w) = \frac{2^{\alpha+\beta}}{p(z,w)(1 - w + p(z,w))^{\alpha}(1 + w + p(z,w))^{\beta}} \tag{1.10}$$

$$= \frac{2^{\alpha+\beta}}{\sqrt{1 - 2zw + w^2}(1 - w + \sqrt{1 - 2zw + w^2})^{\alpha}(1 + w + \sqrt{1 - 2zw + w^2})^{\beta}}$$

and assume that $P^{(\alpha,\beta)}(z,0) \equiv 1$.

(II.1.2) *For $z \in H$ and $w \in U(0; |\omega(z)|^{-1})$ it holds*

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(z) w^n = P^{(\alpha,\beta)}(z,w). \tag{1.11}$$

Proof. $P^{(\alpha,\beta)}(z,w)$, as a function of w, is holomorphic in the disk $U(0; |\omega(z)|^{-1})$ and, hence, by Taylor's theorem it has a representation as a power series centered at the origin, i.e.

$$P^{(\alpha,\beta)}(z,w) = \sum_{n=0}^{\infty} a_n^{(\alpha,\beta)}(z) w^n. \tag{1.12}$$

If $0 < r < |\omega(z)|^{-1}$, then for the coefficients in (1.12) we have

$$a_n^{(\alpha,\beta)}(z) = \frac{1}{2\pi i} \int_{C(0;r)} \frac{P^{(\alpha,\beta)}(z,w)}{w^{n+1}}\, dw,\ n = 0, 1, 2, \ldots.$$

From $p^2(z,w) = 1 - 2zw + w^2$ it follows that $p'_w(z,0) = -z$, and using (1.7) we find that $\zeta'(0) = -1 + z^2 \neq 0$ for $z \in H$. Hence, there exists a neighbourhood $U(0;\delta)$ with $0 < \delta < |\omega(z)|^{-1}$, where the function $\zeta(w)$ is univalent. Since $\zeta(0) = z$, it is clear that for arbitrary $r \in (0,\delta)$ the image $\gamma(z;r)$ of the circle $C(0;r)$ by the map $\zeta(w)$ is a positively oriented rectifiable Jordan curve with the property that $\mathrm{ind}(\gamma(z;r), z) = 1$. Moreover, r can be chosen such that $\gamma(z;r) \cup \mathrm{int}\gamma(z;r) \subset H \cap S(1;z) \cap S(-1;z)$.

Using the representation (1.5) with $\gamma = \gamma(z;r)$ and the equalities

$$\frac{\zeta^2(w) - 1}{2(\zeta(w) - z)} = \frac{1}{w},\ \frac{\zeta'(w)}{\zeta(w) - z} = \frac{1}{wp(z,w)},\ w \in U(0; |\omega(z)|^{-1})$$

and denoting the variable ζ by w, we obtain

$$P_n^{(\alpha,\beta)}(z) = \frac{1}{2\pi i}\int_{C(0;r)} \frac{P^{(\alpha,\beta)}(z,w)}{w^{n+1}}\, dw,\ n = 0,1,2,\ldots.$$

Hence, $a_n^{(\alpha,\beta)}(z) = P_n^{(\alpha,\beta)}(z)$ for $n = 0,1,2,\ldots$. Then from (1.12) it follows that (1.11) holds in the disk $U(0;|\omega(z)|^{-1})$.

Let g be the unique complex function which is holomorphic in the region $G = \mathbb{C} \setminus \{(-\infty,-1] \cup [1,\infty)\}$, and such that $g^2(z) = 1 - z^2$ and $g(0) = 1$. The function $\tau(z)$ defined as $\tau(z) = z + ig(z)$ is holomorphic and nowhere vanishing in G, and, moreover, $(\tau(z) + (\tau(z))^{-1})/2 = z$ for $z \in G$. Hence, $\tau(z)$ is an inverse of the Zhukovskii function $z = (\tau + (\tau)^{-1}))/2$ and as such, it is univalent in the half-plane $\operatorname{Im}\tau > 0$ and maps it onto the region G. In particular, the image of the point i is the origin. More precisely, the image of the halfplane $\operatorname{Im} z > 0$ is the region determined by the inequalities $|\tau| < 1$ and $\operatorname{Im}\tau > 0$, while the image of the interval $(-1,1)$ is the arc of the unit circle located in the half-plane $\operatorname{Im}\tau > 0$. The image of the half-plane $\operatorname{Im} z < 0$ is the region determined by $|\tau| > 1$ and $\operatorname{Im}\tau > 0$.

The proof of **(II.1.2)** leads to the following assertion:

(II.1.3) *The equality* (1.11) *holds for arbitrary* $z \in G$ *and* $w \in U(0;\rho(z))$, *where* $\rho(z) = \min(|\tau(z)|, |\tau(z)|^{-1})$.

A particular case of (1.11) is the representation

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) w^n = P^{(\alpha,\beta)}(x,w), \tag{1.13}$$

which holds for $-1 < x < 1$ and $|w| < 1$. Indeed, in this case we have $\tau(x) = |x + i\sqrt{1-x^2}| = 1$.

2. Integral representation and generating functions for Laguerre polynomials and associated functions

2.1 Here we intend to discuss the most familiar integral representations and generating functions for the (classical) Laguerre polynomials.

(II.2.1) *If* $r \in (0,1)$ *and* $\alpha \in \mathbb{C} \setminus \mathbb{Z}^-$, *then for* $z \in \mathbb{C}$ *and* $n = 0,1,2,\ldots$,

$$L_n^{(\alpha)}(z) = \frac{1}{2\pi i}\int_{C(0;r)} w^{-n-1}(1-w)^{-1-\alpha} \exp\Big(-\frac{zw}{1-w}\Big)\, dw. \tag{2.1}$$

Proof. Since both sides of (2.1) are complex-valued functions holomorphic in the whole complex plane, we need to verify the validity of (2.1) only for $z = x > 0$.

Suppose that $x \in (0, \infty)$ and let $\sigma \subset \mathbb{C} \setminus (-\infty, 0]$ be an arbitrary positively oriented rectifiable Jordan curve such that $\operatorname{ind}(\sigma; x) = 1$. Then from [Chapter I, (3.1)] we obtain

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha} \exp x}{2\pi i} \int_\sigma \frac{\zeta^{n+\alpha} \exp(-\zeta)}{(\zeta - x)^{n+1}} \, d\zeta \tag{2.2}$$

for $n = 0, 1, 2, \dots$

The image of the unit disk $U(0;1)$ under the homographic transformation $\zeta = \zeta(w) = x(1-w)^{-1}$ is the half-plane $T : \operatorname{Re}\zeta > x/2$. Since $\zeta(0) = x$, the image of any circle $C(0;r)$, where $r \in (0,1)$, is a positively oriented circle $\sigma(r)$ such that $\operatorname{ind}(\sigma(r); x) = 1$. Since $(x(1-w)^{-1})^{n+\alpha} = x^{n+\alpha}(1-w)^{-n-\alpha}$, from (2.2) with $\sigma = \sigma(r)$, denoting ζ by w, we obtain

$$L_n^{(\alpha)}(x) = \frac{1}{2\pi i} \int_{C(0;r)} w^{-n-1}(1-w)^{-1-\alpha} \exp\Big(-\frac{xw}{1-w}\Big) \, dw, \ n = 0, 1, 2, \dots.$$

Consider the function

$$L^{(\alpha)}(z, w) = (1-w)^{-1-\alpha} \exp\Big(-\frac{zw}{1-w}\Big), \tag{2.3}$$

for $z \in \mathbb{C}, w \in U(0;1)$ and $\alpha \in \mathbb{C}$. Then, the equalities (2.1) and the Taylor theorem lead to the following proposition:

(II.2.2) *For $\alpha \in \mathbb{C} \setminus \mathbb{Z}^-, z \in \mathbb{C}$ and $w \in U(0;1)$, the identity*

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(z) w^n = L^{(\alpha)}(z, w). \tag{2.4}$$

holds true.

For $z \neq 0$ we define the function $B_\alpha(z)$ by

$$B_\alpha(z) = z^{-\alpha/2} J_\alpha(2\sqrt{z}), \tag{2.5}$$

where J_α is the Bessel function of the first kind with index α [Appendix, (2.2)] assuming that $B_\alpha(0) = (\Gamma(\alpha+1))^{-1}$.

From [Appendix, (2.2)] it follows that if $\alpha \in \mathbb{C}$ is fixed, then $B_\alpha(z)$ is an entire function of the complex variable z. Its Taylor representation centered at the origin is

$$B_\alpha(z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu z^\nu}{\nu! \Gamma(\nu + \alpha + 1)} \cdot . \tag{2.6}$$

There is a generating function for the Laguerre polynomials involving the entire function B_α. More precisely, the following assertion is true:

(II.2.3) *If $\alpha \in \mathbb{C} \setminus \mathbb{Z}^-$, then*

$$\sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(z)}{\Gamma(n+\alpha+1)} w^n = \exp w B_\alpha(zw). \tag{2.7}$$

for $z \in \mathbb{C}$ and $w \in \mathbb{C}$.

Proof. From (2.6) and the Cauchy rule for multiplication of power series it follows that

$$\exp w B_\alpha(zw) = \sum_{n=0}^{\infty} \Big\{ \sum_{\nu=0}^{n} \frac{(-1)^\nu z^\nu}{\nu!(n-\nu)!\Gamma(\nu+\alpha+1)} \Big\}.$$

Since

$$\binom{n+\alpha}{n-\nu} = \frac{\Gamma(n+\alpha+1)}{(n-\nu)!\Gamma(\nu+\alpha+1)}$$

for $n = 0, 1, 2, \ldots$ and $\nu = 0, 1, 2, \ldots$, [I,(2.8)] gives

$$\sum_{\nu=0}^{n} \frac{(-1)^\nu z^\nu}{\nu!(n-\nu)!\Gamma(\nu+\alpha+1)} = \frac{L_n^{(\alpha)}(z)}{\Gamma(n+\alpha+1)}, \; n = 0, 1, 2, \ldots.$$

If $z \in \mathbb{C}$ is fixed, then $B_\alpha(z)$ is an entire function of α. Indeed, each term of the series in the right-hand side of (2.6) is an entire function of α. Moreover, this series is absolutely uniformly convergent on each compact subset of the complex plane. In order to prove the last property one can use the well-known geometric test due to D'Alembert. If we denote $u_\nu(z,\alpha) = (-1)^\nu z^\nu (\nu!\Gamma(\nu+\alpha+1))^{-1}$, then

$$\lim_{\nu\to\infty} \Big| \frac{u_{\nu+1}(z,\alpha)}{u_\nu(z,\alpha)} \Big| = 0$$

uniformly with respect to α in any disk $U(0;r)$ with $r \in \mathbb{R}^+$.

(II.2.4) *For $n = 0, 1, 2, \ldots$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > -1-n$, the integral*

$$I_n(z,\alpha) = \int_0^n t^{n+\alpha} \exp(-t) B_\alpha(zt)\, dt \tag{2.8}$$

is absolutely uniformly convergent with respect to z on every compact subset of $\mathbb{C}$. Moreover, for $z \in \mathbb{C}$ and $\sigma \in (0,1)$, an inequality of the kind

$$|I_n(z,\alpha)| \le \operatorname{Const}(z,\sigma)\sigma^{-n}\Gamma(n+\operatorname{Re}\alpha+1), \; n = 0, 1, 2, \ldots \tag{2.9}$$

holds.

Proof. By means of the coefficients of the power series representation (2.6) it can be proved that the order of $B_\alpha(z)$ as an entire function of z is equal to $1/2$. Hence, there exists a constant $B \in \mathbb{R}^+$ such that for $z \in \mathbb{C}$

$$|B_\alpha(z)| \le B \exp(|z|^{3/4}). \tag{2.10}$$

Therefore, for $r \in \mathbb{R}^+$ the integral

$$\int_0^\infty t^{n+\operatorname{Re}\alpha} \exp(-t)|B_\alpha(zt)|\, dt \tag{2.11}$$

is majorized on the disk $U(0;r)$ by the convergent integral

$$\int_0^\infty t^{n+\operatorname{Re}\alpha} \exp(-t + r^{3/4}t^{3/4})\, dt.$$

If $z \in \mathbb{C}$ and $\sigma \in (0,1)$ are fixed, then the function $\exp[-(1-\sigma)t + |z|^{3/4}t^{3/4}]$ is bounded on the interval $[0,\infty)$. Then, using (2.10), we find

$$|I_n(z,\alpha)| \le \operatorname{Const}(z,\sigma) \int_0^\infty t^{n+\operatorname{Re}\alpha} \exp(-\sigma t)\, dt$$

$$= \operatorname{Const}(z,\sigma)\sigma^{-n}\Gamma(n + \operatorname{Re}\alpha + 1)$$

for $n = 0, 1, 2, \ldots$.

(II.2.5) *If $z \in \mathbb{C}$ and $n = 0, 1, 2, \ldots$ are fixed, then the integral in* (2.8) *is absolutely convergent with respect to α on every compact subset of the half-plane* $\operatorname{Re}\alpha > -1 - n$.

Proof. From the integral representation [Appendix, (2.15)] it follows that

$$B_\alpha(zt) = \frac{1}{2\pi i} \int_{L(\delta,\rho)} \zeta^{-1-\alpha} \exp(\zeta - zt/\zeta)\, d\zeta. \tag{2.12}$$

If $z \ne 0$, then let us take $\rho = 2|z|$. Since $|\zeta| \ge \rho$ for $\zeta \in L(\delta,\rho)$, we have $|zt/\zeta| \le t/2$ whenever $0 \le t < \infty$ and $\zeta \in L(\delta, 2|z|)$. Then (2.12) yields

$$|B_\alpha(zt)| \le \frac{\exp(t/2)}{2\pi} \int_{L(\delta,2|z|)} |\zeta^{-1-\alpha} \exp\zeta|\, ds.$$

The above inequality implies the majorization of the integral in (2.8) on every compact subset of the half-plane $\operatorname{Re}\alpha > -1 - n$ by an integral of the kind

$$\int_0^\infty t^\lambda \exp(-t/2)\, dt$$

with $\lambda > -1$.

(II.2.6) *For $z \in \mathbb{C}$, $n = 0, 1, 2, \ldots$ and $\alpha \in \mathbb{C}$ with* $\operatorname{Re}\alpha > -1 - n$ *it holds*

$$L_n^{(\alpha)}(z) = \frac{\exp z}{n!} \int_0^\infty t^{n+\alpha} \exp(-t) B_\alpha(zt)\, dt. \tag{2.13}$$

Proof. If n and z are fixed, then $L_n^{(\alpha)}(z)$ is an entire function of α. Since **(II.2.5)** implies that the right-hand side of (2.13) is a holomorphic function of α in the half-plane $\operatorname{Re}\alpha > -1-n$, it is sufficient to prove the validity of (2.13) only when $\operatorname{Re}\alpha > -1$. Suppose that this requirement is fulfilled.

The particular case of (2.13) which corresponds to $n = 0$, i.e. the equality

$$\exp(-z) = \int_0^\infty t^\alpha \exp(-t) B_\alpha(zt)\, dt,$$

can be proved by replacing z by zt in the right-hand side of (2.5) followed by multiplying by $t^\alpha \exp(-t)$ and termwise integrating on the interval $(0, \infty)$. The last "operation" is available since after the integration we obtain a series which is absolutely convergent for $z \in \mathbb{C}$.

For $z \in \mathbb{C}$ the power series

$$\sum_{n=0}^\infty \frac{I_n(z,\alpha)}{n!} w^n \tag{2.14}$$

is convergent in the disk $U(0;1)$. Indeed, if $\sigma \in (0,1)$, then the inequality (2.9) and the Stirling formula yield that $\limsup_{n\to\infty} |I_n(z,\alpha)(n!)^{-1}|^{1/n} \le \sigma^{-1}$. Hence, the function $\tilde{L}^{(\alpha)}(z,w)$ defined for $z \in \mathbb{C}$ and $w \in U(0;1)$ by

$$\tilde{L}^{(\alpha)}(z,w) = \sum_{n=0}^\infty \frac{w^n}{n!} \int_0^\infty t^{n+\alpha} \exp(-t) B_\alpha(zt)\, dt \tag{2.15}$$

is holomorphic as a function of w in the disk $U(0;1)$. In fact,

$$\tilde{L}^{(\alpha)}(z,w) = \int_0^\infty \Big\{ \sum_{n=0}^\infty \frac{(wt)^n}{n!} \Big\} t^\alpha \exp(-t) B_\alpha(zt)\, dt,$$

i.e.

$$\tilde{L}^{(\alpha)}(z,w) = \int_0^\infty t^\alpha \exp[-(1-w)t] B_\alpha(zt)\, dt. \tag{2.16}$$

Both sides in (2.16) are holomorphic functions of w in the disk $U(0;1)$. Since from (2.14) and (2.3) it follows that

$$\int_0^\infty t^\alpha \exp[-(1-u)t] B_\alpha(zt)\, dt = (1-u)^{-1-\alpha} \int_0^\infty t^\alpha \exp(-t) B_\alpha(z(1-u)^{-1}t)\, dt$$

$$= (1-u)^{-1-\alpha} \exp\Big(-\frac{z}{1-u}\Big) = \exp(-z) L^{(\alpha)}(z,u)$$

for $u \in (-1,1)$, the identity theorem implies that the equality $\tilde{L}^{(\alpha)}(z,w)$

$= \exp(-z)L^{(\alpha)}(z,w)$ holds whenever $z \in \mathbb{C}$ and $w \in U(0;1)$. Then, the validity of (2.13) for each $z \in \mathbb{C}$ and $n = 0,1,2,\dots$ follows from (2.4) and (2.15).

We have already mentioned that the Hermite polynomials can be expressed by means of the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ with $\alpha = \pm 1/2$ [Chapter I, (5.24), (5.25)]. Conversely, there exist integral representations of the Laguerre polynomials in terms of the Hermite polynomials:

(II.2.7) *For* $z \in \mathbb{C}$ *and* $\alpha \in \mathbb{C}$, *with* $\operatorname{Re}\alpha > -1/2$,

$$L_n^{(\alpha)}(z^2) = \frac{2(-1)^n\Gamma(n+\alpha+1)}{\sqrt{\pi}(2n)!\Gamma(\alpha+1/2)} \int_0^1 (1-t^2)^{\alpha-1/2} H_{2n}(zt)\,dt,\ n = 0,1,2,\dots. \tag{2.17}$$

Proof. From [Chapter I, (2.12)] it follows that

$$H_{2n}(zt) = (-1)^n(2n)! \sum_{k=0}^{n} \frac{(-1)^k(2zt)^{2k}}{(2k)!(n-k)!},\ n = 0,1,2,\dots.$$

Since $\operatorname{Re}\alpha > -1/2$, we have

$$\int_0^1 (1-t^2)^{\alpha-1/2} t^{2k}\,dt = \frac{1}{2}\int_0^1 (1-u)^{\alpha-1/2}u^{k-1/2}\,du$$

$$= \frac{\Gamma(\alpha+1/2)\Gamma(k+\alpha/2)}{2\Gamma(k+\alpha+1)} = (2k-1)(2k-3)\dots 3.1\sqrt{\pi}\Gamma(\alpha+1/2),\ k = 0,1,2,\dots.$$

But $(2k)! = k!2^k(2k-1)(2k-3)\dots 3.1$ and, hence,

$$\frac{2(-1)^n\Gamma(n+\alpha+1)}{\sqrt{\pi}(2n)!\Gamma(\alpha+1/2)} \int_0^1 (1-t^2)^{\alpha-1/2} H_{2n}(zt)\,dt$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{k!}\binom{n+\alpha}{n-k} z^{2k} = L_n^{(\alpha)}(z^2),\ n = 0,1,2,\dots.$$

2.2 There are several integral representations of the Laguerre associated functions depending on their domains of existence.

(II.2.8) *If* $\operatorname{Re} z < 0$ *and* $\operatorname{Re}\alpha > -1$, *then*

$$M_n^{(\alpha)}(z) = -(-z)^{\alpha} \int_0^{\infty} \frac{t^{n+\alpha}\exp(zt)}{(1+t)^{n+1}}\,dt \tag{2.18}$$

for $n = 0,1,2,\dots$.

Proof. If $\operatorname{Re} z < 0$ and $r > 0$, then let $C(z,r)$ be the circular arc lying in right half-plane and having r as its initial point and $r\exp i\theta$, where $\theta = \arg(-z)$, as its endpoint. Then

$$\lim_{r\to\infty}\int_{C(z,r)}\frac{\zeta^{n+\alpha}\exp(-\zeta)}{(\zeta-z)^{n+1}}\,d\zeta,\ n=0,1,2,\dots. \tag{2.19}$$

Indeed, if $\zeta\in C(z,r)$, then $|\exp(-\zeta)|\le\exp(-r\cos\theta)$ and, hence,

$$\Big|\int_{C(z,r)}\frac{\zeta^{n+\alpha}\exp(-\zeta)}{(\zeta-z)^{n+1}}\,d\zeta\Big|\le\frac{\pi r^{n+\operatorname{Re}\alpha+1}\exp(-r\cos\theta)}{2(r-|z|)^{n+1}}$$

for $r>|z|$.

Furthermore,

$$\Big|\int_{C(z,\delta)}\frac{\zeta^{n+\alpha}\exp(-\zeta)}{(\zeta-z)^{n+1}}\,d\zeta\Big|\le\frac{\pi\delta^{n+\operatorname{Re}\alpha+1}}{2(|z|-\delta)^{n+1}},$$

and, due to $\operatorname{Re}\alpha>-1$, we obtain that

$$\lim_{\delta\to 0}\int_{C(z,\delta)}\frac{\zeta^{n+\alpha}\exp(-\zeta)}{(\zeta-z)^{n+1}}\,d\zeta=0,\ n=0,1,2,\dots. \tag{2.20}$$

Denote by $C^*(z,\delta)$ the circular arc lying in the right half-plane and having $\delta\exp i\theta$ and δ as its initial and its endpoint, respectively. If $\delta<|z|<r$, then denote by $L(z,\delta,r)$ the path consisting of the segment $[\delta,r]$, the arc $C(z,r)$, the segment $[r\exp i\theta,\delta\exp i\theta]$, and the arc $C^*(z,\delta)$. Then, according to the Cauchy integral theorem

$$\int_{L(z,\delta,r)}\frac{\zeta^{n+\alpha}\exp(-\zeta)}{(\zeta-z)^{n+1}}\,d\zeta=0,\ n=0,1,2,\dots. \tag{2.21}$$

If $l(-z):\zeta=(-z)t, 0\le t<\infty$, then using the integral representation [I, (4.13)] as well as the equalities (2.19), (2.20) and (2.21), we obtain

$$M_n^{(\alpha)}(z)=-\int_{l(-z)}\frac{\zeta^{n+\alpha}\exp(-\zeta)}{(\zeta-z)^{n+1}}\,d\zeta,\ n=0,1,2,\dots.$$

Setting $\zeta=(-z)t, 0\le t<\infty$, we arrive to the integral representation (2.18).

Let us now define

$$M^{(\alpha)}(z,w)=-(-z)^{\alpha}\int_0^{\infty}\frac{t^{\alpha}}{1+t}\exp\Big\{zt+\frac{wt}{1+t}\Big\}\,dt, \tag{2.22}$$

for $\operatorname{Re}z<0,\operatorname{Re}\alpha>-1$ and $w\in\mathbb{C}$.

(II.2.9) *If* $\operatorname{Re}z<0,\operatorname{Re}\alpha>-1$ *and* $w\in\mathbb{C}$, *then*

$$\sum_{n=0}^{\infty}\frac{M_n^{(\alpha)}(z)}{n!}w^n=M^{(\alpha)}(z,w). \tag{2.23}$$

Proof. Since $0 \le t/(1+t) < 1$ for $t \in [0,\infty)$, (2.18) implies

$$|M_n^{(\alpha)}(z)| \le |(-z)^\alpha| \int_0^\infty t^{\operatorname{Re}\alpha} \exp(xt)\, dt,\ \ x = \operatorname{Re} z,$$

and, hence, the power series on the left-hand side of (2.23) converges in the whole complex plane, provided $\operatorname{Re} z < 0$. Therefore, if $w \in \mathbb{C}$, then the series

$$\sum_{n=0}^\infty \frac{(-z)^\alpha}{n!}\left\{\frac{wt}{1+t}\right\}^n t^\alpha \exp(zt)$$

can be integrated term by term on the interval $(0,\infty)$ with respect to the variable t. Thus, we obtain the equality (2.23).

As a corollary of [Chapter I, (5.21)] and the integral representation [Appendix, (3.13)] of the Tricomi function we obtain the following proposition:

(II.2.10) *If* $\operatorname{Re}\alpha > -1$, *then for* $z \in \mathbb{C} \setminus [0,\infty)$ *and* n=0, 1, 2, ...

$$M_n^{(\alpha)}(z) = -2(-z)^{\alpha/2} \int_0^\infty t^{n+\alpha/2} \exp(-t) K_\alpha(2\sqrt{-zt})\, dt, \tag{2.24}$$

where K_α *is the modified Bessel function of the third kind with index* α.

3. Integral representations and generating functions for Hermite polynomials and associated functions

3.1 Having in mind the definition of the Hermite polynomials given in [Chapter I, **2.3**] as well as the integral representation [Chapter I, (1.12)], we find that for $r \in \mathbb{R}^+$

$$H_n(z) = \frac{(-1)^n n!}{2\pi i} \int_{C(0;r)} \frac{\exp(-\zeta^2 - 2z\zeta)}{\zeta^{n+1}}\, d\zeta,\ \ n = 0,1,2,\ldots.$$

Replacing z by $-z$ and using the relation $H_n(-z) = (-1)^n H_n(z)$, $\quad n = 0,1,$ $2,\ldots$, we arrive to

$$H_n(z) = \frac{n!}{2\pi i} \int_{C(0;r)} \frac{\exp(-\zeta^2 + 2z\zeta)}{\zeta^{n+1}}\, d\zeta,\ \ n = 0,1,2,\ldots. \tag{3.1}$$

Further, from the above integral representations it follows that for $z \in \mathbb{C}$ and $w \in \mathbb{C}$ it holds the identity

$$\sum_{n=0}^\infty \frac{H_n(z)}{n!} w^n = \exp(-w^2 + 2zw). \tag{3.2}$$

(II.3.1) *If $z \in \mathbb{C}$ and $\delta \in \mathbb{R}^+$, then there exists a positive integer $n_0 = n_0(z, \delta)$ such that*

$$|H_n(z)| \leq (2n/e)^{n/2}(1+\delta)^n \tag{3.3}$$

for $n \geq n_0$.

Proof. If z is fixed, then $\exp(-w^2 + 2zw)$ is an entire function of order 2 and type 1. Therefore, the equality $\limsup_{n\to\infty} n^{1/2}\{|H_n(z)|(n!)^{-1}\}^{1/n} = \sqrt{2e}$ holds. Hence, for each $\delta \in \mathbb{R}^+$ there exists a positive integer $n_0 = n_0(z, \delta)$ such that $|H_n(z)| \leq n!(2e/n)^{n/2}(1+\delta)^{n/2}$ for $n \geq n_0$. Since n_0 can be chosen such that $n! \leq (n/e)^{n/2}(1+\delta)^{n/2}$ for $n \geq n_0$, we conclude that inequality (3.3) holds for $n \geq n_0$.

The relation [Chapter I, (5.16)] and the integral representations [Appendix, (4.5)] of the Weber-Hermite functions yield

$$H_n(z) = \frac{2^{n+1}}{\sqrt{\pi}} \exp z^2 \int_0^\infty t^n \exp(-t^2) \cos(2zt - n\pi/2)\, dt \tag{3.4}$$

whenever $z \in \mathbb{C}$ and $n = 0, 1, 2, \ldots$. An immediate consequences of the above equalities are the following integral representations of the Hermite polynomials

$$H_n(z) = \frac{2^n(-i)^n}{n!} \exp z^2 \int_{-\infty}^\infty t^n \exp(-t^2 + 2izt)\, dt, \; n = 0, 1, 2, \ldots. \tag{3.5}$$

(II.3.2) *If $z \in \mathbb{C}, \zeta \in \mathbb{C}$ and $w \in U(;1)$, then*

$$\sum_{n=0}^\infty \frac{H_n(z)H_n(\zeta)}{n!2^n} w^n = (1-w^2)^{-1/2} \exp\Big\{\frac{2z\zeta w - (z^2 + \zeta^2)w^2}{1-w^2}\Big\}. \tag{3.6}$$

Proof. If z and ζ are fixed, then the series on the left-hand side of (3.6) is convergent in the unit disk. If $0 < |w| < 1$, then we choose $\delta \in \mathbb{R}^+$ such that $q = (1+\delta)^2|w| < 1$. Since $n! \geq (n/e)^n$ when $n \geq 1$, from **(II.3.1)** it follows that there exists a positive integer $n_0 = n_0(z, \zeta, \delta)$ such that

$$\Big|\frac{H_n(z)H_n(\zeta)w^n}{n!2^n}\Big| \leq q^n$$

whenever $n \geq n_0$. Obviously, the above inequality holds for $w = 0$ and $n = 0, 1, 2, \ldots$.

Further, both (3.4) and the representation

$$H_n(\zeta) = \frac{2^n(-i)^n}{\sqrt{\pi}} \exp \zeta^2 \int_{-\infty}^\infty \tau^n \exp(-\tau^2 + 2i\zeta\tau)\, d\tau,$$

show that for $w \in \mathbb{C}$ and $n = 0, 1, 2, \dots$ it holds

$$\frac{H_n(z)H_n(\zeta)w^n}{n!2^n} \tag{3.7}$$
$$= \frac{\exp(z^2+\zeta^2)}{\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(-2t\tau w)^n}{n!} \exp(-t^2-\tau^2-2izt-2i\zeta\tau)\,dt\,d\tau.$$

Consider the series whose general term is the integrand of the double integral in (3.7). If $w \in U(0;1)$, then this series can be termwise integrated on $\mathbb{R}^2$ and, hence, we can write

$$\sum_{n=0}^{\infty} \frac{H_n(z)H_n(\zeta)}{n!2^n} w^n \tag{3.8}$$
$$= \frac{\exp(z^2+\zeta^2)}{\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-t^2-\tau^2-2wt\tau-2izt-2i\zeta\tau)\,dt\,d\tau.$$

Since

$$\int_{-\infty}^{\infty} \exp(-a^2s^2-2bs)\,ds = \frac{\sqrt{\pi}}{a}\exp(b^2/a^2) \tag{3.9}$$

when $\operatorname{Re} a^2 > 0$, we find that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-t^2-\tau^2-2wt\tau-2izt-2i\zeta\tau)\,dt\,d\tau$$
$$= \int_{-\infty}^{\infty} \exp(-t^2-2izt)\,dt \int_{-\infty}^{\infty} \exp[-\tau^2-2(wt+i\zeta)\tau]\,d\tau$$
$$= \sqrt{\pi}\exp(-\zeta^2)\int_{-\infty}^{\infty} \exp[-(1-w^2)t^2-2i(z-w\zeta)t]\,dt$$
$$= \pi(1-w^2)^{-1/2}\exp\Big\{-\zeta^2-\frac{z^2-2z\zeta w+\zeta^2w^2}{1-w^2}\Big\},$$

thus, proving (3.6).

3.2 We leave the proof of the following proposition as an exercise to the reader:

(II.3.3) *The equalities*

$$(\mathbf{a})\ \sum_{n=0}^{\infty} \frac{G_n^+(z)}{n!} w^n = G_0^+(z-w), \tag{3.10}$$

and

$$(\mathbf{b})\ \sum_{n=0}^{\infty} \frac{G_n^-(z)}{n!} w^n = G_0^-(z-w) \tag{3.11}$$

hold true when $z \in \mathbb{C}$ and $w \in \mathbb{C}$.

Exercises

1. Verify the validity of equality (2.7) for $\alpha = -k,\ k = 1, 2, 3, \dots$.

2. Prove that

$$\sqrt{\pi} H_n(z) = 2^n \int_{-\infty}^{\infty} (z + it)^n \exp(-t^2)\, dt$$

for $z \in \mathbb{C}$ and $n = 0, 1, 2, \dots$

3. If $\operatorname{Re} \lambda > 0$, then prove the validity of the integral representation

$$P_n^{(\lambda)}(z) = \frac{\Gamma(n + 2\lambda)}{n! 2^{2\lambda - 1} (\Gamma(\lambda))^2} \int_0^{\pi} (z + \sqrt{z^2 - 1} \cos \varphi)^n (\sin \varphi)^{2\lambda - 1}\, d\varphi,$$

$$z \in \mathbb{C},\ n = 0, 1, 2, \dots$$

4. Prove that for $z \in \mathbb{C}$

$$P_n(z) = \pi^{-1} \int_0^{\pi} (z + \sqrt{z^2 - 1} \cos \varphi)^n\, d\varphi,\ n = 0, 1, 2, \dots,$$

$$P_n(z) = \pi^{-1} \int_0^{\pi} (z + \sqrt{z^2 - 1} \cos \varphi)^{-n-1}\, d\varphi,\ n = 0, 1, 2, \dots.$$

Remark. The above representations of Legendre polynomials are known as the first and second Laplace integrals, respectively.

5. Prove that if $z \in \mathbb{C} \setminus [-1, 1], w \in U(0; |\omega(z)|^{-1})$ and $2\lambda \neq 0, -1, -2, \dots,$ then

$$(\mathbf{a})\ \sum_{n=0}^{\infty} \frac{\Gamma(n + \lambda + 1/2)}{\Gamma(n + 2\lambda)} P_n^{(\lambda)}(z) w^n = P^{(\lambda)}(z, w),$$

where

$$P^{(\lambda)}(z, w) = \frac{2^{\lambda - 1/2} \Gamma(\lambda + 1/2)}{\Gamma(2\lambda) \sqrt{1 - 2zw + w^2} (1 - zw + \sqrt{1 - 2zw + w^2})^{\lambda - 1/2}}$$

6. Prove that **(a)** holds when $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty]\}, w \in U(0; \rho(z))$ and $2\lambda \neq 0, -1, -2, \dots$. In particular, (**a**) holds when $z = x \in (-1, 1)$ and $|w| < 1$.

7. Prove that if $z \in \mathbb{C} \setminus [-1, 1], w \in U(0; |\omega(z)|^{-1})$ and $\lambda \neq 0, -1, -2, \dots$ then

$$(\mathbf{b})\ \sum_{n=0}^{\infty} P_n^{(\lambda)}(z) w^n = (1 - 2zw + w^2)^{-\lambda}.$$

8. Prove that **(b)** holds when $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}, w \in U(0; \rho(z))$ and $\lambda \neq 0, -1, -2, \dots$. In particular, (**b**) is valid when $z = x \in (-1, 1)$ and $|w| < 1$.

9. Prove that if $z \in \mathbb{C} \setminus [-1, 1]$ and $w \in U(0; |\omega(z)|^{-1})$, then

$$(\mathbf{c})\ 1 + 2\sum_{n=1}^{\infty} T_n(z)w^n = \frac{1 - z^2}{1 - 2zw + w^2},$$

$$(\mathbf{d})\ \sum_{n=0}^{\infty} U_n(z)w^n = \frac{1}{1 - 2zw + w^2}$$

10. Prove that the equalities **(c)** and **(d)** hold for $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$ and $w \in U(0; \rho(z))$. In particular, they hold for $z = x \in (-1, 1)$ and $|w| < 1$.

11. Suppose that $\operatorname{Re}\alpha > -1, \operatorname{Re}\beta > -1$ and define the function

$$K^{(\alpha,\beta)}(z, w) = \int_{-1}^{1} \frac{(1-t)^{\alpha}(1+t)^{\beta}}{z - t} \exp\left\{\frac{(1-t^2)w}{2(z-t)}\right\} dt$$

for $z \in \overline{\mathbb{C}} \setminus [-1, 1]$ and $w \in \mathbb{C}$. Prove that for such z and w,

$$\sum_{n=0}^{\infty} Q_n^{(\alpha,\beta)}(z)\frac{w^n}{n!} = K^{(\alpha,\beta)}(z, w).$$

Comments and references

All integral representations and generating functions for the Jacobi and Laguerre polynomials and associated functions included in this chapter are established under the most general conditions on the parameters α and β. In some cases they may be arbitrary complex numbers.

The equality (2.7) is due to G. Doetsch [H. Bateman, A. Erdélyi, 1, 10.12, (18)] and the integral representation (2.13) is a corollary of [Appendix, (3.12)] *via* [I, (6.4)]. We prefer to give an independent proof of (2.13) as well as to use the entire function $B_\alpha(z)$ instead of the Bessel function $J_\alpha(z)$ of the first kind with index α for which the origin could be a branch point.

The integral representation (2.17) of the Laguerre polynomials by means of even Hermite polynomials is known as a formula of J. V. Uspensky [1]. It plays a considerable role in Chapter V, where the representation of holomorphic functions as series in Laguerre polynomials is discussed.

The equality (3.6) is an example of the so called bilinear generating functions. It is due to F.G. Mehler [1].

Chapter III

ASYMPTOTIC FORMULAS. INEQUALITIES

1. Asymptotic formulas for Jacobi polynomials and associated functions

1.1 Recall that the function $\omega(z) = z + \sqrt{z^2-1}$ [Chapter II, (1.6)] is meromorphic in the subdomain $\overline{\mathbb{C}} \setminus [-1,1]$ of the extended complex plane and that it has a simple pole at the point of infinity. For $z \in \mathbb{C} \setminus [-1,1]$ we define $\bar{l}(z) = \{\zeta = (1+t)\omega(z),\ 0 \le t < \infty\} \cup \{\infty\}$ and assume that $\bar{l}(\infty) = \infty$. The function ψ of the complex variables z, w, defined in the region $A = \{(z,w) : z \in \overline{\mathbb{C}} \setminus [-1,1], w \in \overline{\mathbb{C}} \setminus \bar{l}(z)\} \subset \overline{\mathbb{C}} \times \overline{\mathbb{C}}$ by $\psi(z,w) = \exp\{(1/2)\log(1 - w/\omega^2(z))\}$, is holomorphic in A, and $\psi^2(z,w) = 1 - w/\omega^2(z), \psi(z,0) \equiv 1$. The function φ, defined in the region $\mathbb{C} \setminus [1,\infty)$ by $\varphi(w) = \exp\{(1/2)\log(1-w)\}$, is holomorphic in this region, and $\varphi^2(w) = 1 - w,\ \varphi(0) = 1$.

If B is the region $\{\overline{\mathbb{C}} \setminus [-1,1]\} \times \{\mathbb{C} \setminus [1,\infty)\}$, then the intersection $D = A \cap B$ is also a region in $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$. Moreover, if $p(z,w)$ is the function in the right-hand side of [Chapter II, (1.10)], then the function $p(z, w/\omega(z))$ is holomorphic in D and $p(z, w/\omega(z)) = \psi(z,w)\varphi(w) = \sqrt{1 - w/\omega^2(z)}\sqrt{1-w}$. Hence, the function

$$A^{(\alpha,\beta)}(z,w) = \sqrt{1-w}P^{(\alpha,\beta)}(z, w/\omega(z)), \tag{1.1}$$

where $P^{(\alpha,\beta)}$ is the function defined by [Chapter II, (1.10)], is also holomorphic in the region D. This function can be analytically continued in the region A as a holomorphic function of the complex variables z, w. In particular, if $z \in \overline{\mathbb{C}} \setminus [-1,1]$ is fixed, then $A^{(\alpha,\beta)}(z,w)$ is holomorphic in the disk $U(1; |\omega(z)|^2 - 1)$. By Taylor's theorem its power series representation in a neighbourhood of $w = 1$ has the form

$$A^{(\alpha,\beta)}(z,w) = \sum_{k=0}^{\infty} A_k^{(\alpha,\beta)}(z)(1-w)^k. \tag{1.2}$$

Suppose that $r \in (1,\infty)$ and denote by $E(r)$ the interior of the ellipse $e(r) = \{z \in \mathbb{C} : |\omega(z)| = r\}$. If $0 < \rho < r^2 - 1$, then for $z \in \overline{\mathbb{C}} \setminus \overline{E(r)}$ and $k = 0, 1, 2, \ldots$

$$A_k^{(\alpha,\beta)}(z) = \frac{1}{2\pi i} \int_{C(1;\rho)} \frac{A^{(\alpha,\beta)}}{(1-w)^{k+1}}\, dw. \tag{1.3}$$

From the above representation it follows that each of the functions $A_k^{(\alpha,\beta)}(z)$, $k = 0, 1, 2, \ldots$, is holomorphic in the region $\overline{\mathbb{C}} \setminus \overline{E(r)}$. The union of these regions as r runs the interval $(1,\infty)$ is the region $\overline{\mathbb{C}} \setminus [-1,1]$ and, hence, each of the functions (1.3), $k = 0, 1, 2, \ldots$ is holomorphic in the region $\overline{\mathbb{C}} \setminus [-1,1]$.

Consider

$$P_\nu^{(\alpha,\beta)}(z,w) = P^{(\alpha,\beta)}(z, w/\omega(z)) - \sum_{k=0}^{\nu} A_k^{(\alpha,\beta)}(z)(1-w)^{k-1/2}, \tag{1.4}$$

where $(z,w) \in S = \{\overline{\mathbb{C}} \setminus [-1,1]\} \times \{\mathbb{C} \setminus [1,\infty)\}$ and let

$$P_\nu^{(\alpha,\beta)}(z,w) = \sum_{n=0}^{\infty} A_{\nu,n}^{(\alpha,\beta)}(z) w^n \tag{1.5}$$

for $w \in U(0;1)$.

(III.1.1) *If $1 < r < \infty, \nu = 0, 1, 2, \dots,$ then the sequence $\{n^{\nu+1} A_{\nu,n}^{(\alpha,\beta)}(z)\}_{n=1}^{\infty}$ is uniformly bounded on $K(r) = \overline{\mathbb{C}} \setminus E(r)$.*

Proof. The representation

$$P_\nu^{(\alpha,\beta)}(z,w) = \sum_{k=\nu+1}^{\infty} A_k^{(\alpha,\beta)}(z)(1-w)^{k-1/2}$$

holds for $w \in U(0;1) \cap U(1; |\omega(z)|^2 - 1)$. Then it is clear that the function

$$\varphi_\nu^{(\alpha,\beta)}(z,w) = \frac{\partial^{\nu+1} P_\nu^{(\alpha,\beta)}(z,w)}{\partial w^{\nu+1}}$$

is holomorphic in the region S and that it can be extended as a continuous function of $(z,w) \in \{\overline{\mathbb{C}} \setminus [-1,1]\} \times \overline{U(0;1)}$. From (1.5) it follows that for $n > \nu$

$$n(n-1)(n-2)\dots(n-\nu) A_{\nu,n}^{(\alpha,\beta)}(z) = \frac{1}{2\pi i} \int_{C(0;1)} \frac{\varphi_\nu^{(\alpha,\beta)}(z,w)}{w^{n-\nu}\sqrt{1-w}}\, dw. \tag{1.6}$$

Denoting by $\gamma_\rho, 0 < \rho < \min(1, |\omega(z)|^2 - 1)$, the arc of the circle $C(1;\rho)$ lying in the disk $U(0;1)$, and by C_ρ the arc of $C(0;1)$ which is outside the disk $U(1;\rho)$, then for $n > \nu$ we have

$$n(n-1)(n-2)\dots(n-\nu) A_{\nu,n}^{(\alpha,\beta)}(z)$$

$$= \frac{1}{2\pi i} \int_{C_\rho} \frac{\varphi_\nu^{(\alpha,\beta)}(z,w)}{w^{n-\nu}\sqrt{1-w}}\, dw - \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{\varphi_\nu^{(\alpha,\beta)}(z,w)}{w^{n-\nu}\sqrt{1-w}}\, dw.$$

Since

$$\lim_{\rho \to 0} \int_{C_\rho} \frac{\varphi_\nu^{(\alpha,\beta)}(z,w)}{w^{n-\nu}\sqrt{1-w}}\, dw = \int_{C(0;1)} \frac{\varphi_\nu^{(\alpha,\beta)}(z,w)}{w^{n-\nu}\sqrt{1-w}}\, dw$$

and

$$\lim_{\rho \to 0} \int_{\gamma_\rho} \frac{\varphi_\nu^{(\alpha,\beta)}(z,w)}{w^{n-\nu}\sqrt{1-w}}\, dw = 0,$$

we conclude that the equality (1.6) holds for $z \in \overline{\mathbb{C}} \setminus [-1,1]$ and $n > \nu$.

Since the function $|\varphi_\nu^{(\alpha,\beta)}(z,w)|$ is bounded on $\{\overline{\mathbb{C}} \setminus E(r)\} \times \overline{U(0;1)}$, where $1 < r < \infty$, from the equalities (1.6) it follows that the sequence $\{n^{\nu+1}A_{\nu,n}^{(\alpha,\beta)}(z)\}_{n=1}^{\infty}$ is uniformly bounded on $K(r)$, where $1 < r < \infty$.

(III.1.2) *The representation*

$$(1.7)\qquad (\omega(z))^{-n}P_n^{(\alpha,\beta)}(z)$$
$$= \sum_{k=0}^{\nu} \frac{(-1)^k\Gamma(k+1/2)\Gamma(n-k+1/2)}{\pi\Gamma(n+1)} A_k^{(\alpha,\beta)}(z) + A_{\nu,n}^{(\alpha,\beta)}(z)$$

holds for $z \in \overline{\mathbb{C}} \setminus [-1,1], \nu = 0,1,2,\dots$ *and* $n = 0,1,2,\dots$.

Proof. Taking into account [Chapter II, (1.11)], we find that

$$A_{\nu,n}^{(\alpha,\beta)}(z) = (\omega(z))^{-n}P_n^{(\alpha,\beta)}(z) - (-1)^n \sum_{k=0}^{\nu} \binom{k-1/2}{n} A_k^{(\alpha,\beta)}(z)$$

and, hence,

$$(\omega(z))^{-n}P_n^{(\alpha,\beta)}(z) = (-1)^n \sum_{k=0}^{\nu} \binom{k-1/2}{n} A_k^{(\alpha,\beta)}(z) + A_{\nu,n}^{(\alpha,\beta)}(z)$$

for $z \in \overline{\mathbb{C}} \setminus [-1,1],\ \nu = 0,1,2,\dots$, and $n = 0,1,2,\dots$.

The equality
$$\binom{a}{n} = \frac{(-1)^n\Gamma(n-a)}{n!\Gamma(-a)},$$
which is valid for $a \neq 0,1,2,\dots$, as well as the relation $\Gamma(-k+1/2)\Gamma(k+1/2) = (-1)^k\pi,\ k = 0,1,2,\dots$, yield that

$$\binom{k-1/2}{n} = \frac{(-1)^{n+k}\Gamma(k+1/2)\Gamma(n-k+1/2)}{\pi\Gamma(n+1)},$$

and, thus, we obtain the representation (1.7).

For $\nu = 0$, as a corollary of (1.7), we obtain

$$(1.8)\qquad (\omega(z))^{-n}P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(n+1/2)}{\sqrt{\pi}\Gamma(n+1)} A_0^{(\alpha,\beta)}(z) + A_{0,n}^{(\alpha,\beta)}(z)$$

and, moreover, that the sequence $\{nA_{0,n}^{(\alpha,\beta)}(z)\}_{n=1}^{\infty}$ is uniformly bounded on $K(r)$ when $1 < r < \infty$.

From (1.1) and (1.2) it follows that

$$A_0^{(\alpha,\beta)}(z) = A^{(\alpha,\beta)}(z,1) = \{\sqrt{1-w}P^{(\alpha,\beta)}(z, w/\omega(z))\}_{w=1}$$

for $z \in \overline{\mathbb{C}} \setminus [-1,1]$ and, hence, $A_0^{(\alpha,\beta)}(z) \neq 0$ for each $z \in \overline{\mathbb{C}} \setminus [-1,1]$. Then, (1.8) as well as a particular case of [Appendix, (1.13)], lead to the following proposition:

(III.1.3). *If $z \in \overline{\mathbb{C}} \setminus [-1,1]$, then*

$$(1.9) \qquad P_n^{(\alpha,\beta)}(z) = \pi^{-1/2} n^{-1/2} (\omega(z))^n p^{(\alpha,\beta)}(z)\{1 + p_n^{(\alpha,\beta)}(z)\},\ n = 1,2,3,\dots,$$

where the complex functions $p^{(\alpha,\beta)}(z)$, $p_n^{(\alpha,\beta)}(z)$, $n = 1,2,3,\dots$, are holomorphic in the region $\overline{\mathbb{C}} \setminus [-1,1]$. Moreover, $p^{(\alpha,\beta)}(z) \neq 0$ for every z in this region, and the sequence $\{\sqrt{n} p_n^{(\alpha,\beta)}(z)\}_{n=1}^{\infty}$ is uniformly bounded on $K(r)$, where $1 < r < \infty$.

1.2 We need the asymptotics of Jacobi associated functions $\{Q_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ when n tends to infinity and z ranges on an arbitrary closed subset of the region $\overline{\mathbb{C}} \setminus [-1,1]$. The final result is an asymptotic formula which is a corollary of some auxiliary propositions.

(III.1.4) *If $\alpha+1, \beta+1$ and $\alpha+\beta+2$ are not equal to $0,-1,-2,\dots$, then for $z \in \overline{\mathbb{C}} \setminus [-1,-1]$*

$$(1.10) \qquad (\omega(z))^{n+1} Q_n^{(\alpha,\beta)}(z) = \sum_{k=0}^{\infty} B_{n,k}^{(\alpha,\beta)} (\omega(z))^{-k},\ n = 0,1,2,\dots,$$

where the coefficients satisfy the recurrence relation

$$(1.11) \qquad (k+1)(2n+k+\alpha+\beta+2)B_{n,k+1} + 2(\alpha-\beta)(n+k+1)B_{n,k}$$
$$+(\alpha+\beta-k)(2n+k)B_{n,k-1} = 0,\ k = 1,2,3,\dots.$$

For $k = 0,1,2,\dots$ there exists

$$(1.12) \qquad B_k^{(\alpha,\beta)} = (2\sqrt{\pi})^{-1} \lim_{n\to\infty} n^{1/2} B_{n,k}^{(\alpha,\beta)}$$

and the following recurrence relation

$$(1.13) \qquad (k+1)B_{k+1}^{(\alpha,\beta)} + (\alpha-\beta)B_k^{(\alpha,\beta)} + (\alpha+\beta-k)B_{k-1}^{(\alpha,\beta)} = 0,\ k = 1,2,3,\dots$$

holds.

Proof. Since the function $Q_n^{(\alpha,\beta)}$ is holomorphic in the domain $\overline{\mathbb{C}} \setminus [-1,1]$, and since it has a zero of multiplicity $n+1$ at the point ∞, the function

$$(1.14) \qquad B_n^{(\alpha,\beta)}(\zeta) = \zeta^{-n-1} Q_n^{(\alpha,\beta)}((\zeta + \zeta^{-1})/2),\ n = 0,1,2,\dots$$

is holomorphic in the unit disk $U(0;1)$. Let

$$(1.15) \qquad B_n^{(\alpha,\beta)}(\zeta) = \sum_{k=0}^{\infty} B_{n,k}^{(\alpha,\beta)} \zeta^k.$$

be its power series expansion in $U(0;1)$. Setting $\zeta = (\omega(z))^{-1}$, where $z \in \overline{\mathbb{C}}\backslash[-1,1]$, we obtain the representation (1.10).

From [Chapter I, (5.17)] it follows that if $\zeta \in U(0;1)$ and $n = 0,1,2,\dots$, then

$$(1.16)\qquad \frac{\Gamma(2n+\alpha+\beta+2)}{2^{2n+\alpha+\beta+2}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}B_n^{(\alpha,\beta)}(\zeta)$$

$$= (1-\zeta)^{-2n-2}F(n+\alpha+1, n+1, 2n+\alpha+\beta+2; -4\zeta/(1-\zeta)^2)).$$

Since the function $F(n+\alpha+1, n+1, 2n+\alpha+\beta+2; z)$ is a solution of the differential equation

$$z(1-z)w'' + [2n+\alpha+\beta+2-(2n+\alpha+3)z]w' - (n+1)(n+\alpha+1)w = 0,$$

it is easy to verify that the function $B_n^{(\alpha,\beta)}(\zeta)$ satisfies the equation

$$\zeta(1-\zeta^2)w'' + [(\alpha+\beta-2n-4)\zeta^2 + 2(\alpha-\beta)\zeta + \alpha+\beta+2n+2]w'$$

$$+2(n+1)[(\alpha+\beta-1)\zeta+\alpha-\beta]w = 0.$$

Then, by inserting the power series from (1.15) into the last equation, followed by equating the coefficient of $\zeta^{k+1}, k \ge 0$, to zero, we get the relation (1.11) as well as the equality

$$(1.17)\qquad (2n+\alpha+\beta+2)B_{n,1}^{(\alpha,\beta)} + 2(n+1)(\alpha-\beta)B_{n,0}^{(\alpha,\beta)} = 0.$$

If $\alpha+1, \beta+1$ and $\alpha+\beta+2$ are not equal to $0,-1,-2,\dots$, then

$$(1.18)\qquad B_{n,0} = \frac{2^{2n+\alpha+\beta+2}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)},\ n = 0,1,2,\dots.$$

Since $B_{n,0}^{(\alpha,\beta)} = B_n^{(\alpha,\beta)}(0)$, the above equality is a direct consequence of (1.16). Using it together with the Stirling formula, we find that

$$(1.19)\qquad \lim_{n\to\infty} n^{1/2}B_{n,0}^{(\alpha,\beta)} = 2\sqrt{\pi}$$

and then (1.17) implies

$$(1.20)\qquad \lim_{n\to\infty} n^{1/2}B_{n,1}^{(\alpha,\beta)} = 2\sqrt{\pi}(\beta-\alpha).$$

Observe that the recurrence relation (1.11) can be rewritten in the form

$$(1.21)\qquad B_{n,k+1}^{(\alpha,\beta)} = -\frac{2(\alpha-\beta)(n+k+1)}{(k+1)(2n+\alpha+\beta+2)}B_{n,k}^{(\alpha,\beta)}$$

$$-\frac{(\alpha+\beta-k)(2n+k+1)}{(k+1)(2n+\alpha+\beta+2)}B_{n,k-1}^{(\alpha,\beta)},\ k = 1,2,3,\dots.$$

Then from (1.19), (1.20) and (1.21) it follows by induction that for $k = 2, 3, 4 \dots$ there exists $\lim_{n\to\infty} n^{1/2} B_{n,k}$. In particular, using (1.12) as well as (1.19) and (1.20) we find that $B_0^{(\alpha,\beta)} = 1$, and $B_1^{(\alpha,\beta)} = \beta - \alpha$. Then (1.21) yields the relation (1.13).

Further, define the function

$$q^{(\alpha,\beta)}(z) = (1 - (\omega(z))^{-1})^{\alpha-1/2}(1 + (\omega(z))^{-1})^{\beta-1/2}. \tag{1.22}$$

in the region $\overline{\mathbb{C}} \setminus [-1, 1]$.

(III.1.5) *The representation*

$$q^{(\alpha,\beta)}(z) = \sum_{k=0}^{\infty} B_k^{(\alpha,\beta)}(\omega(z))^{-k} \tag{1.23}$$

holds in the region $\overline{\mathbb{C}} \setminus [-1, 1]$.

Proof. For $\zeta \in U(0; 1)$ we consider the function

$$B(\alpha, \beta; \zeta) = q^{(\alpha,\beta)}((\zeta + \zeta^{-1})/2) = (1 - \zeta)^{\alpha-1/2}(1 + \zeta)^{\beta-1/2}.$$

By Taylor's theorem,

$$B(\alpha, \beta; \zeta) = \sum_{k=0}^{\infty} \tilde{B}_k^{(\alpha,\beta)}\zeta^k, \zeta \in U(0; 1).$$

A direct verification shows that if $\zeta \in U(0; 1)$, then

$$(1 - \zeta^2)B'(\alpha, \beta; \zeta) + [(\alpha + \beta - 1)\zeta + \alpha - \beta]B(\alpha, \beta; \zeta) = 0.$$

Therefore, the coefficients $\{\tilde{B}_k^{(\alpha,\beta)}\}_{k=0}^{\infty}$ satisfy the same recurrence relation (1.13) as the numbers $\{B_k^{(\alpha,\beta)}\}_{k=0}^{\infty}$ do. Since $\tilde{B}_0^{\alpha,\beta} = B(\alpha, \beta; 0) = 1 = B_0^{(\alpha,\beta)}$ and $\tilde{B}_1^{(\alpha,\beta)} = B'(\alpha, \beta; 0) = \beta - \alpha = B_1^{(\alpha,\beta)}$, it follows that $\tilde{B}_k^{(\alpha,\beta)} = B_k^{(\alpha,\beta)}$ for $k = 0, 1, 2, \dots$. Thus, (1.23) is proved.

(III.1.6) *Let* $\alpha + 1, \beta + 1$ *and* $\alpha + \beta + 2$ *be not equal to* $0, -1, -2, \dots$. *Then for each* $r \in (1, \infty)$,

$$\lim_{n\to\infty} (2\sqrt{\pi})^{-1} n^{1/2}(\omega(z))^{n+1}Q_n^{(\alpha,\beta)}(z) = q^{(\alpha,\beta)}(z) \tag{1.24}$$

uniformly on $\overline{\mathbb{C}} \setminus E(r)$

Proof. Define

$$M_k^{(\alpha,\beta)} = \sup_{n\geq 1}(2\sqrt{\pi})^{-1} n^{1/2}|B_{n,k}^{(\alpha,\beta)}|, \ k = 0, 1, 2, \dots. \tag{1.25}$$

Since each of the sequences $\{n^{1/2}B_{n,k}^{(\alpha,\beta)}\}_{n=1}^{\infty}$, $k = 0,1,2\dots$ is convergent, we have that $M_k^{(\alpha,\beta)} < \infty$ for $k = 0,1,2,\dots$. More precisely,

$$\limsup_{k\to\infty}\{M_k^{(\alpha,\beta)}\}^{1/k} \le 1. \tag{1.26}$$

Denote $\lambda = |\alpha-\beta|, \mu = |\alpha+\beta|$ and $k_0 = 1+\max(0,[\mu])$. Since

$$(2n+k+1)/(2n+k-\mu) \le (k+1)/(k-\mu)$$

for $n = 0,1,2,\dots$ and $k \ge k_0$, as a corollary of (1.21) we obtain that the inequality

$$\left|B_{n,k+1}^{(\alpha,\beta)}\right| \le \frac{k+1}{k-\mu}\left\{\frac{2\lambda}{k+1}\left|B_{n,k}^{(\alpha,\beta)}\right| + \frac{k+\mu}{k+1}\left|B_{n,k-1}^{(\alpha,\beta)}\right|\right\} \tag{1.27}$$

holds for $n = 0,1,2,\dots$ and $k \ge k_0$.

Define the function $\sigma(\lambda,\mu;\zeta) = (1-\zeta)^{-(2\lambda+\mu+1)/2}(1+\zeta)^{(2\lambda-\mu-1)/2}$ in the region $\mathbb{C}\setminus\{(-\infty,-1]\cup[1,\infty)\}$. Let

$$\sigma(\lambda,\mu;\zeta) = \sum_{k=0}^{\infty}\sigma_k^{(\lambda,\mu)}\zeta^k$$

be its Taylor's expansion in the disk $U(0;1)$. It is easy to verify that the function $\sigma(\lambda,\mu;\zeta)$ satisfies the differential equation $(1-\zeta^2)w' - [(\mu+1)\zeta + 2\lambda]w = 0$ and that the last property leads to the recurrence relation

$$\sigma_{k+1}^{(\lambda,\mu)} = \frac{2\lambda}{k+1}\sigma_k^{(\lambda,\mu)} + \frac{k+\mu}{k+1}\sigma_{k-1}^{(\lambda,\mu)},\ k = 1,2,3,\dots. \tag{1.28}$$

Since $\sigma_0^{(\lambda,\mu)} = \sigma(\lambda,\mu;0) = 1$, and $\sigma_1^{(\lambda,\mu)} = \sigma'(\lambda,\mu;0) = 2\lambda = 2|\alpha-\beta|$, from (1.28) it follows that all the coefficients $\sigma_k^{(\lambda,\mu)}, k = 0,1,2,\dots$, are nonnegative. Moreover, if $\alpha \ne \beta$, then all they are positive. If $\alpha = \beta$, then the function $\sigma(0,\mu;\zeta)$ is even and, hence, in this case $\sigma_{2k+1}^{(\lambda,\mu)} = 0, k = 0,1,2,\dots$, and $\sigma_{2k}^{(\lambda,\mu)} > 0$, $k = 0,1,2,\dots$.

The positive constant L can be chosen such that

$$(2\sqrt{\pi})^{-1}n^{1/2}\left|B_{n,k_0}^{(\alpha,\beta)}\right| \le L\sigma_{k_0}^{(\lambda,\mu)}$$

and

$$(2\sqrt{\pi})^{-1}n^{1/2}\left|B_{n,k_0-1}^{(\alpha,\beta)}\right| \le L\sigma_{k_0-1}^{(\lambda,\mu)}$$

for $n = 1,2,3,\dots$. Obviously, if $\alpha \ne \beta$, then this is possible, since any of the sequences $\{n^{1/2}B_{n,k}^{(\alpha,\beta)}\}_{n=1}^{\infty}, k = 0,1,2,\dots$, is convergent and, moreover, $\sigma_k^{(\lambda,\mu)} > 0, k = 0,1,2,\dots$. If $\alpha = \beta$, then from (1.14) and the relation $Q_n^{(\alpha,\beta)}(-z)$

$= (-1)^{n+1}Q_n^{(\beta,\alpha)}(z), n = 0,1,2,\ldots,$ [Chapter I, Exercise 24] it follows that for $n = 0,1,2,\ldots$ the function $B_n^{(\alpha,\alpha)}(\zeta)$ is even, hence, $B_{n,2k+1}^{(\alpha,\alpha)} = 0, n = 0,1,2,\ldots,k = 0,1,2,\ldots.$ Let us recall that in this case $\sigma_{2k+1}^{(0,\mu)} = 0, k = 0,1,2,\ldots.$ The inequalities (1.27) and the recurrence relation (1.28) give that if $k \geq k_0$, then

$$(2\sqrt{\pi})^{-1}n^{1/2}\left|B_{n,k+1}^{(\alpha,\beta)}\right| \leq L\sigma_{k+1}^{(\lambda,\mu)}\prod_{s=k_0}^{k}\frac{s+1}{s-\mu},$$

i.e.

$$(2\sqrt{\pi})^{-1}n^{1/2}\left|B_{n,k+1}^{(\alpha,\beta)}\right| \leq L\sigma_{k+1}^{(\lambda,\mu)}\frac{\Gamma(k_0-\mu)\Gamma(k+1)}{\Gamma(k_0+1)\Gamma(k-\mu)}.$$

Hence,

$$M_{k+1}^{(\alpha,\beta)} \leq L\sigma_{k+1}^{(\lambda,\mu)}\frac{\Gamma(k_0-\mu)\Gamma(k+1)}{\Gamma(k_0+1)\Gamma(k-\mu)}$$

and since $\limsup_{k\to\infty}\{\sigma_k^{(\lambda,\mu)}\}^{1/k} \leq 1$, the Stirling formula yields the inequality (1.26). Since $\left|B_k^{(\alpha,\beta)}\right| \leq M_k^{(\alpha,\beta)}$ for $k = 0,1,2,\ldots,$ it follows that

$$\limsup_{k\to\infty}\left|B_k^{(\alpha,\beta)}\right|^{1/k} \leq 1,\ k = 0,1,2,\ldots.$$

For $n = 1,2,3,\ldots$ and $z \in \overline{\mathbb{C}} \setminus [-1,1]$ consider the function

(1.29) $$r_n^{(\alpha,\beta)}(z) = (2\sqrt{\pi})^{-1}n^{1/2}Q_n^{(\alpha\beta)}(z) - q^{(\alpha,\beta)}(z).$$

Let $\varepsilon > 0, r \in (1,\infty)$ and $\delta > 0$ be chosen such that $1+\delta < r$. Then, there exists a positive integer ν such that $((1+\delta)/r)^\nu < \varepsilon, M_k^{(\alpha,\beta)} \leq (1+\delta)^k$ and $\left|B_k^{(\alpha,\beta)}\right| \leq (1+\delta)^k$ for every $k > \nu$. We can also find a positive integer $N = N(\varepsilon)$ such that

$$\left|(2\sqrt{\pi})^{-1}n^{1/2}B_{n,k}^{(\alpha,\beta)} - B_k^{(\alpha,\beta)}\right| < \varepsilon$$

for $k = 0,1,2,\ldots,\nu$ and $n > N$. Then, for $z \in \overline{\mathbb{C}} \setminus E(r) = \{z \in \overline{\mathbb{C}} : |\omega(z)| \geq r\}$ and $n > N$ we have the inequality

$$|r_n^{(\alpha,\beta)}(z)| \leq \sum_{k=0}^{\infty}\left|(2\sqrt{\pi})^{-1}n^{1/2}B_{n,k}^{(\alpha,\beta)} - B_k^{(\alpha,\beta)}\right||\omega(z)|^{-k}$$

$$\leq \varepsilon\sum_{k=0}^{\nu}r^{-k} + 2\sum_{k=\nu+1}^{\infty}((1+\delta)/r)^k,$$

i.e. $|r_n^{(\alpha,\beta)}(z)| < \{r/(r-1) + 2r/(r-1-\delta)\}\varepsilon$. Thus, the proposition **(III.1.6)** is proved.

(III.1.7) *If $\alpha+1, \beta+1$ and $\alpha+\beta+1$ are not equal to $0, -1, -2, \dots$, then for every $n = 1, 2, 3, \dots$ and every $z \in \overline{\mathbb{C}} \setminus [-1, 1]$*

$$Q_n^{(\alpha,\beta)}(z) = 2\sqrt{\pi} n^{-1/2} (\omega(z))^{-n-1} q^{(\alpha,\beta)}(z) \{1 + q_n^{(\alpha,\beta)}(z)\}. \tag{1.30}$$

Moreover, $\lim_{n\to\infty} q_n^{(\alpha,\beta)}(z) = 0$ uniformly on every set $\overline{\mathbb{C}} \setminus E(r)$, where $r \in (1, \infty)$, and, hence, on each closed subset of the region $\overline{\mathbb{C}} \setminus [-1, 1]$.

Proof. From (1.29) it follows that if $z \in \overline{\mathbb{C}} \setminus [-1, 1]$ and $n = 1, 2, 3, \dots$, then

$$Q_n^{(\alpha,\beta)}(z) = 2\sqrt{\pi} n^{-1/2} (\omega(z))^{-n-1} \{q^{(\alpha,\beta)} + r_n^{(\alpha,\beta)}(z)\}.$$

Denoting $q_n^{(\alpha,\beta)}(z) = r_n^{(\alpha,\beta)}(z)/q^{(\alpha,\beta)}(z),\ n = 1, 2, 3, \dots$, we obtain the representation (1.30).

The identity $\Delta_\nu^{(\alpha,\beta)}(z, z) \equiv 1$ [Chapter I, Exercise 27] yields that for $z \in \overline{\mathbb{C}} \setminus [-1, 1]$,

$$\begin{aligned} &P_\nu^{(\alpha,\beta)}(z) Q_{\nu+1}^{(\alpha,\beta)}(z) - P_{\nu+1}^{(\alpha,\beta)}(z) Q_\nu^{(\alpha,\beta)}(z) \\ &= -\frac{2^{\alpha+\beta+1}(2\nu+\alpha+\beta)\Gamma(\nu+\alpha+1)\Gamma(\nu+\beta+1)}{\Gamma(\nu+2)\Gamma(\nu+\alpha+\beta+2)}, \nu = 0, 1, 2, \dots. \end{aligned} \tag{1.31}$$

The above equality as well as the representations (1.9) and (1.30) give

$$R(\alpha, \beta; \nu) := p^{(\alpha,\beta)}(z) q^{(\alpha,\beta)}(z) \{(1 + p_\nu^{(\alpha,\beta)}(z)\}\{1 + q_{\nu+1}^{(\alpha,\beta)}(z)\}$$

$$-(\omega(z))^{-2}\{1 + p_{\nu+1}^{(\alpha,\beta)}(z)\}\{1 + q_\nu^{(\alpha,\beta)}(z)\} = R(\alpha, \beta; \nu)$$

$$= \frac{2^{\alpha+\beta}\nu^{1/2}(\nu+1)^{1/2}(2\nu+\alpha+\beta)\Gamma(\nu+\alpha+1)\Gamma(\nu+\beta+1)}{\Gamma(\nu+2)\Gamma(\nu+\alpha+\beta+2)}.$$

By means of Stirling's formula we find that $\lim_{\nu\to\infty} R(\alpha, \beta; \nu) = 2^{\alpha+\beta+1}$. Thus, we come to the relation $p^{(\alpha,\beta)}(z) q^{(\alpha,\beta)}(z)(1 - (\omega(z))^{-2}) = 2^{\alpha+\beta+1}$, $z \in \overline{\mathbb{C}} \setminus [-1, 1]$. Then, as a direct result of the definition of the function $q^{(\alpha,\beta)}(z)$, we obtain that the equality

$$p^{(\alpha,\beta)}(z) = 2^{\alpha+\beta+1}(1 - (\omega(z))^{-1})^{-\alpha-1/2}(1 + (\omega(z))^{-1})^{-\beta-1/2} \tag{1.32}$$

holds in the region $\overline{\mathbb{C}} \setminus [-1, 1]$.

2. Asymptotic formulas for Hermite and Laguerre polynomials

2.1 The asymptotics of the Hermite polynomials $\{H_n(z)\}_{n=0}^{\infty}$ when z is bounded and n tends to infinity is given by the following proposition:

(III.2.1) *If $z \in \mathbb{C}$, $n = 0, 1, 2, \ldots$ and $s = 1, 2, 3, \ldots$, then*

$$H_n(z) = \lambda_n \exp(z^2/2)\Big\{\cos[(2n+1)^{1/2}z - n\pi/2]\sum_{k=0}^{s-1} u_k(z)(2n+1)^{-k} \tag{2.1}$$

$$+\sin[(2n+1)^{1/2}z - n\pi/2]\sum_{k=0}^{s-1} v_k(z)(2n+1)^{-k-1/2} + h_{s,n}(z)\Big\},$$

$$\lambda_{2n} = \Gamma(2n+1)/\Gamma(n+1),\ \lambda_{2n+1} = (2n+1)^{-1/2}\Gamma(2n+3)/\Gamma(n+2),$$

where $\{u_k(z)\}_{k=0}^{\infty}$ and $\{v_k(z)\}_{k=0}^{\infty}$ are (algebraic) polynomials not depending on n and s. In particular, $u_0(z) \equiv 1$. Moreover, $h_{s,n}(z), s = 1, 2, 3, \ldots; n = 0, 1, 2, \ldots,$ are entire functions of the complex variable z such that for every fixed $s = 1, 2, 3, \ldots$ the sequence $\{n^s \exp[-(2n+1)^{1/2}|\operatorname{Im} z|]h_{s,n}(z)\}_{n=1}^{\infty}$ is uniformly bounded on every compact subset of the complex plane.

The case $s = 1$ yields

$$H_n(z) = \lambda_n \exp(z^2/2)\{\cos[(2n+1)^{1/2}z - n\pi/2] + \tilde{h}_n(z)\},$$

where

$$\tilde{h}_n(z) = (2n+1)^{-1/2}\sin[(2n+1)^{1/2} - n\pi/2] + h_{1,n}(z),\ n = 0, 1, 2, \ldots.$$

Stirling's formula enables us to write $\lambda_n = \sqrt{2}(2n/e)^{n/2}(1 + O(n^{-1})),\ n \to \infty$. Thus, we obtain the representation

$$H_n(z) = \sqrt{2}(2n/e)^{n/2}\exp(z^2/2)\{\cos[(2n+1)^{1/2}z - n\pi/2] + h_n(z)\}, \tag{2.2}$$

where $\{h_n(z)\}_{n=0}^{\infty}$ are entire functions and the sequence

$$\{n^{1/2}\exp[-(2n+1)^{1/2}|\operatorname{Im} z|]h_n(z)\}_{n=1}^{\infty} \tag{2.3}$$

is uniformly bounded on every compact subset of $\mathbb{C}$.

We are going to give an independent proof of (2.2). It is based on the integral representation

$$\pi 2^{-n/2}(\Gamma(n+1))^{-1}H_n(z) \tag{2.4}$$

$$= i^{n+1}\int_{ic-\infty}^{ic+\infty}\exp(-\zeta^2/2)\zeta^{-n-1}\cosh(\sqrt{2}z\zeta - n\pi i/2)\,d\zeta, \quad c > 0,$$

which follows from [Chapter I, (5.16)] and [Appendix, (4.7)].

At first we are to prove the following auxiliary proposition:

(III.2.2) *Define*

(2.5) $$\omega(\xi;t) = t^2/2 + i\xi^{-1}t - \xi^{-2}\log(1+i\xi t)$$

for $\in \mathbb{R}, \xi \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, *and let* $\omega(0;t) \equiv 0$. *Then*:

(a) $|\omega(\xi;t)| \le (1/3)|\xi||t|^3, \quad \xi, t \in \mathbb{R}$;

(b) $\mu = \sup_{\xi,t\in\mathbb{R}} t^{-2}|\omega(\xi;t)| < 1$.

Proof. Since $\omega(\xi;0) = \omega'(\xi;0) = \omega''(\xi;0) = 0$, Taylor's formula with remainder term in an integral form yields

$$\omega(\xi;t) = (t^3/2)\int_0^1 (1-s)^2\omega'''(\xi;ts)\,ds = i\xi t^3 \int_0^1 (1-s)^2(1+i\xi ts)^{-3}\,ds.$$

Hence,

$$|\omega(\xi;t)| \le |\xi||t|^3 \int_0^1 (1-s)^2\,ds = (1/3)|\xi||t|^3,$$

and, thus, (a) is established.

Define $f(\zeta) = 1/2 + \zeta^{-1} - \zeta^{-2}\log(1+\zeta)$ for $\zeta \in \{\mathbb{C} \setminus (-\infty,-1)]\} \setminus \{0\}$ and let $f(0) = 1$. Then (2.5) gives

(2.6) $$\omega(\xi;t) = t^2(1 - f(i\xi t)), \quad \xi, t \in \mathbb{R}.$$

Denote $p(\tau) = \mathrm{Re}(1 - f(i\tau))$ and $q(\tau) = \mathrm{Im}(1 - f(i\tau))$ for $\tau \in \mathbb{R}$. Then $p(\tau) = 1/2 - (1/2\tau^2)\log(1+\tau^2)$ and $q(\tau) = \tau^{-1} - \tau^{-2}\operatorname{arctg}\tau$ for $\tau \in \mathbb{R}^* = \mathbb{R}\setminus\{0\}$.

It is easy to prove that $p(\tau) > 0$ and $p'(\tau) > 0$ for $\tau > 0$. Since $\lim_{\tau\to\infty} p(\tau) = 1/2$ and, moreover, $p(-\tau) = p(\tau)$ for $\tau \in \mathbb{R}$, it follows that $\sup_{\tau\in\mathbb{R}} |p(\tau)| = \sup_{\tau\in\mathbb{R}} p(\tau) = 1/2$.

It is clear that $q(\tau) > 0$ for $\tau > 0$. Since $(1+s^2)^{-1} - 1 + s > 0$ for $s > 0$, it follows that $\displaystyle\int_0^1 ((1+s^2)^{-1} - 1 + s)\,ds > 0$ for $\tau > 0$, i.e., $\operatorname{arctg}\tau > \tau - \tau^2/2$ for $\tau > 0$. It means that $q(\tau) < 1/2$ for $\tau > 0$. Further, since $q(0) = 0$, and $\lim_{\tau\to\infty} q(\tau) = 0$, and, moreover, $q(-\tau) = q(\tau)$ for $\tau \in \mathbb{R}$, it follows that $\sup_{\tau\in\mathbb{R}} |q(\tau)| < 1/2$. Then from (2.6) we obtain

$$|\omega(\xi;t)| \le t^2 \sup_{\xi,t\in\mathbb{R}} |1 - f(i\xi t)| = t^2 \sup_{\tau\in\mathbb{R}} |1 - f(i\tau)|$$

$$\le t^2(\sup_{\tau\in\mathbb{R}} |p(\tau)| + \sup_{\tau\in\mathbb{R}} |b(\tau)|) \le t^2(1/2 + \sup_{\tau\in\mathbb{R}} |q(\tau)|),$$

and, since $\mu \le 1/2 + \sup_{\tau\in\mathbb{R}} |q(\tau)|$, **(b)** is verified.

By choosing $c = (n+1/2)^{1/2}$ and setting $\zeta = -t+i(n+1/2)^{1/2}$, $-\infty < t < \infty$, in the integral on the right-hand side of (2.4), we obtain

$$K_n^{-1} H_n(z)$$

$$= A_n(z)\cos((2n+1)^{1/2}z - n\pi/2) + B_n(z)\sin((2n+1)^{1/2}z - n\pi/2),$$

where

$$K_n = \pi^{-1}2^{n/2}\Gamma(n+1)(n+1/2)^{-(n+1)/2}\exp(n/2+1/4),$$

$$A_n(z) = \int_{-\infty}^{\infty} \exp(-t^2+\sqrt{2}zt)a_n(t)\,dt,$$

$$B_n(z) = \int_{-\infty}^{\infty} \exp(-t^2+\sqrt{2}zt)b_n(t)\,dt,$$

$$a_n(t) = (1/2)(\varphi_n(t)\exp(\omega_n(t)) + \varphi_n(-t)\exp(\omega_n(-t))),$$

$$b_n(t) = (i/2)(\varphi_n(t)\exp(\omega_n(t))) - \varphi(-t)\exp(\omega_n(-t)),$$

$$\varphi_n(t) = (1+i(n+1/2)^{-1/2}t)^{-1/2},\ t \in \mathbb{R},$$

and

$$\omega_n(t) = \omega((n+1/2)^{-1/2};t),\ t \in \mathbb{R}.$$

Since

$$\int_{-\infty}^{\infty} \exp(-t^2+\sqrt{2}zt)\,dt = \sqrt{\pi}\exp(z^2/2), \tag{2.7}$$

we have

$$\pi^{-1/2}\exp(-z^2/2)H_n(z) = \cos((2n+1)^{1/2}z - n\pi/2) + h_n^*(z),$$

where

$$h_n^*(z) = \exp(-z^2/2)\cos((2n+1)^{1/2}z - n\pi/2)A_n^*(z)$$
$$+\exp(-z^2/2)\sin((2n+1)^{1/2}z - n\pi/2)B_n(z),$$

$$A_n^*(z) = \int_{-\infty}^{\infty} \exp(-t^2+\sqrt{2}zt)a_n^*(t)\,dt,$$

and $a_n^*(t) = a_n(t) - 1$.

Define $\varphi_n^*(t) = \varphi_n(t) - 1$ and $\omega_n^*(t) = \exp(\omega_n(t)) - 1$, then $a_n^*(t) = (1/2)(\varphi_n^*(t) + \varphi_n^*(-t) + \omega_n^*(t) + \omega_n^*(-t) + \varphi_n^*(t)\omega_n^*(t) + \varphi_n^*(-t)\omega_n^*(-t))$ and $b_n(t) = (i/2)(\varphi_n^*(t) - \varphi_n^*(-t) + \omega_n^*(t) - \omega_n^*(-t) + \varphi_n^*(t)\omega_n^*(t) - \varphi_n^*(-t)\omega_n^*(-t))$.

Since $\varphi_n(t) = 1 + t\int_0^1 \varphi_n'(ts)\,ds$, we have

$$\varphi_n^*(t) = -(i/2)(n+1/2)^{-1/2}t\int_0^1 \frac{ds}{(1+i(n+1/2)^{-1/2}ts)^{3/2}}. \tag{2.8}$$

Then **(III.2.2)** and the inequality $|\exp\zeta - 1| \le |\zeta|\exp(|\zeta|)$ enable us to obtain the estimates

$$(n+1/2)^{1/2}|a_n^*(t)| = O(\eta(\mu;t))$$

and

$$(n+1/2)^{1/2}|b_n(t)| = O(\eta(\mu;t)),$$

where $\eta(\mu;t) = (1+|t|)^4\exp(\mu t^2)$, uniformly for $t \in \mathbb{R}$ when n tends to infinity.

Further, as a corollary of (2.9), (2.10) and (2.7), we obtain that if n tends to infinity, then

$$|A_n^*(z)| = O\left((n+1/2)^{-1/2}\int_{-\infty}^{\infty}(1+|t|)^4\exp(-(1-\mu)t^2 + \sqrt{2}R|t|)\,dt\right)$$

and

$$|B_n(z)| = O\left((n+1/2)^{-1/2}\int_{-\infty}^{\infty}(1+|t|)^4\exp(-(1-\mu)t^2 + \sqrt{2}R|t|)\,dt\right)$$

uniformly for z such that $|z| \le R$ provided R is an arbitrary fixed positive number.

A simple calculation based on Stirling's formula gives

$$K_n = \sqrt{2}\pi^{-1/2}(2n/e)^{n/2}(1+k_n),$$

where $k_n = O(n^{-1})$ when $n \to \infty$. If we define

$$h_n(z) = k_n\cos((2n+1)^{1/2}z - n\pi/2) + (1+k_n)h_n^*(z),$$

then the representation (2.2) follows and, moreover, the sequence (2.3) is uniformly bounded on every compact subset of $\mathbb{C}$.

Remark. Both sides of equality (2.7) are entire functions and, hence, it is sufficient to verify its validity only for $z = x \in \mathbb{R}$. But in this case we have

$$\int_{-\infty}^{\infty}\exp(-t^2+\sqrt{2}xt)\,dt = \exp(x^2/2)\int_{-\infty}^{\infty}\exp(-(t-x/\sqrt{2})^2)\,dt$$

$$= \exp(x^2/2)\int_{-\infty}^{\infty}\exp(-t^2)\,dt = \sqrt{\pi}\exp(x^2/2).$$

2.3 Now we are going to discuss the asymptotics of Laguerre polynomials. We start with the following proposition:

(III.2.3) *If $\alpha \in \mathbb{R}$, then the representation*

(2.9) $$2\sqrt{\pi}(-z)^{\alpha/2+1/4}\exp(-z/2)L_n^{(\alpha)}(z)$$

$$= n^{\alpha/2-1/4}\exp[2(-z)^{1/2}\sqrt{n}]\Big\{\sum_{k=0}^{\nu-1} l_k^{(\alpha)}(z)n^{-k/2} + l_{\nu,n}^{(\alpha)}(z)\Big\},\ n = 1,2,3,\dots$$

holds for the Laguerre polynomials with parameter α in the region $\mathbb{C}\setminus[0,\infty)$. The complex functions $\{l_k^{(\alpha)}(z)\}_{k=0}^{\infty}$ and $\{l_{\nu,n}^{(\alpha)}(z)\}_{\nu,n=1}^{\infty}$ are holomorphic in the region $\mathbb{C}\setminus[0,\infty)$, and, in particular, $l_0^{(\alpha)}(z)\equiv 1$. Moreover, for each fixed $\nu = 1,2,3,\dots$, the sequence $\{n^{\nu/2}l_{\nu,n}^{(\alpha)}\}_{n=1}^{\infty}$ is uniformly bounded on every compact subset of this region.

If $\alpha\in\mathbb{R}\setminus\mathbb{Z}^-$, then the asymptotic formula (2.9) is a particular case of Perron's formula [Appendix, (3.14)], of the relation [Chapter I, (5.12)] as well as of the fact that if $\alpha\in\mathbb{C}\setminus\mathbb{Z}^-$, then

$$\binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)} = \frac{n^\alpha}{\Gamma(\alpha+1)}\Big\{\sum_{k=0}^{\nu-1}\gamma_k^{(\alpha)}n^k + \gamma_{\nu,n}^{(\alpha)}n^{-\nu}\Big\},\ n = 1,2,3,\dots,$$

where the sequence $\{\gamma_{\nu,n}^{(\alpha)}\}_{n=1}^{\infty}$ is bounded for each fixed $\nu = 1,2,3,\dots$ [Appendix, (1.9)].

The validity of (2.9) for $\alpha = -k,\ k = 1,2,3,\dots$, can be verified by means of the relation $L_n^{(-k)}(z) = (-z)^k(n-k)!(n!)^{-1}L_{n-k}^{(k)}(z),\ n = k,k+1,k+2,\dots$ [I, Exercise 8].

Using (2.9), we find the following representation of Laguerre polynomials in the region $\mathbb{C}\setminus[0,\infty)$

$$2\sqrt{\pi}(-z)^{\alpha/2+1/4}\exp(-z/2)L_n^{(\alpha)}(z) \tag{2.10}$$

$$= n^{\alpha/2-1/4}\exp[2(-z)^{1/2}\sqrt{n}]\{1+\lambda_n^{(\alpha)}(z)\},\ n = 1,2,3,\dots.$$

Here $\lambda_n^{(\alpha)}(z) = l_{1,n}^{(\alpha)}(z),\ n = 1,2,3,\dots$, and $\lambda_n^{(\alpha)}(z) = O(n^{-1/2})$ uniformly on every compact subset of the region $\mathbb{C}\setminus[0,\infty)$.

Now we are going to prove an asymptotic formula of the kind (2.10) even in the case when α is an arbitrary complex number.

If $z\in\mathbb{C}\setminus[0,\infty)$ and $n\in\mathbb{N}$ are fixed, then we can extend $\lambda_n^{(\alpha)}(z)$ as a function of α to the whole complex plane by means of the equality

$$\lambda_n^{(\alpha)}(z) = 2\sqrt{\pi}(-z)^{\alpha/2+1/4}n^{-\alpha/2+1/4}\exp[-z/2-2(-z)^{1/2}\sqrt{n}]L_n^{(\alpha)}(z) - 1,$$

Thus, roughly speaking, (2.10) becomes valid for every $\alpha\in\mathbb{C}$ and it remains to study the behaviour of $\lambda_n^{(\alpha)}(z)$ when n tends to infinity.

(III.2.4) *If $\alpha\in\mathbb{C}$ is fixed, then*

$$\lim_{n\to\infty}\lambda_n^{(\alpha)}(z) = 0 \tag{2.11}$$

uniformly on every compact subset of the region $\mathbb{C}\setminus[0,\infty)$.

Proof. If $z\in\mathbb{C}\setminus[0,\infty)$, then $(i(-z)^{1/2})^2=z$ and the integral representation [II, (2.17)] gives for $n=0,1,2,\dots$

$$\text{(2.12)}\qquad \frac{\sqrt{\pi}\Gamma(\alpha+1/2)\Gamma(n+1)}{\Gamma(n+\alpha+1)}\exp[-(4n+1)^{1/2}(-z)^{1/2}]L_n^{(\alpha)}(z)$$

$$=\int_0^1(1-t^2)^{\alpha-1/2}\exp(zt^2/2)\{\exp[-(4n+1)^{1/2}(-z)^{1/2}(1-t)]$$

$$+\exp[-(4n+1)^{1/2}(-z)^{1/2}(1+t)]+\exp[-(4n+1)^{1/2}(-z)^{1/2}]\tilde{h}_{2n}(i(-z)^{1/2}t)\}\,dt.$$

Remark. The above equality holds when $\operatorname{Re}\alpha>-1/2$ but further on we will use it only for $\operatorname{Re}\alpha\ge 1/2$.

We define $\tilde{h}_n(z,t)=\exp[-(4n+1)^{1/2}\operatorname{Re}(-z)^{1/2}t]\tilde{h}_{2n}(i(-z)^{1/2}t)$ provided $z\in\mathbb{C}\setminus[0,\infty), t\in[0,1]$ and $n=0,1,2,\dots$. Since $\operatorname{Re}(-z)^{1/2}>0$ when $z\in\mathbb{C}\setminus[0,\infty)$, $|\operatorname{Im}(i(-z)^{1/2}t)|=\operatorname{Re}((-z)^{1/2}t)$, $t\ge 0$. Therefore, if $M\subset\mathbb{C}\setminus[0,\infty)$ is a compact set, then $\lim_{n\to\infty}\tilde{h}_n(z,t)=0$ uniformly with respect to $(z,t)\in M\times[0,1]$.

Using (2.12), we find

$$\text{(2.13)}\qquad L_n^{(\alpha)}(z)=\frac{\Gamma(n+\alpha+1)\exp[(4n+1)^{1/2}(-z)^{1/2}]}{\sqrt{\pi}\Gamma(\alpha+1/2)\Gamma(n+1)}\sum_{j=1}^{3}K_{n,j}^{(\alpha)}(z),$$

where

$$K_{n,1}^{(\alpha)}(z)=\int_0^1(1-t^2)^{\alpha-1/2}\exp(zt^2/2)\exp[-(4n+1)^{1/2}(-z)^{1/2}(1-t)]\,dt,$$

$$K_{n,2}^{(\alpha)}(z)=\int_0^1(1-t^2)^{\alpha-1/2}\exp(zt^2/2)\exp[-(4n+1)^{1/2}(-z)^{1/2}(1+t)]\,dt$$

and

$$K_{n,3}^{(\alpha)}(z)=\int_0^1(1-t^2)\alpha-1/2\exp(zt^2)\exp[-(4n+1)^{1/2}(-z)^{1/2}$$

$$+(4n+1)^{1/2}\operatorname{Re}(-z)^{1/2}t]\tilde{h}_n(z,t)\,dt.$$

In the integral for $K_{n,1}^{(\alpha)}(z)$ we substitute $(4n+1)^{-1/2}u$ for $1-t$. Thus, we arrive to

$$T_n^{(\alpha)}(z):=\exp(-z/2)(4n+1)^{\alpha/2+1/4}K_{n,1}^{(\alpha)}(z)$$

$$=\int_0^{(4n+1)^{1/2}}u^{\alpha-1/2}R_n^{(\alpha)}(u,z)\exp[-(-z)^{1/2}u]\,du,$$

where

$$R_n^{(\alpha)}(u,z)=[2-(4n+1)^{-1/2}u]^{\alpha-1/2}\exp\{z[-(4n+1)^{-1/2}+(4n+1)^{-1}u^2/2]\}.$$

As a next step we shall prove that

$$\lim_{n\to\infty} T_n^{(\alpha)}(z) = 2^{\alpha-1/2}\Gamma(\alpha+1/2)(-z)^{-\alpha/2-1/4} \tag{2.14}$$

uniformly on every compact subset of the region $\mathbb{C}\setminus[0,\infty)$.

If $\delta, r \in \mathbb{R}^+$, then denote by $D(\delta, r)$ the compact subset of $\mathbb{C}\setminus[0,\infty)$ defined by the inequalities $\operatorname{Re}(-z)^{1/2} \geq \delta$ and $|z| \leq r$. Since each compact subset of $\mathbb{C}\setminus[0,\infty)$ is contained in some $D(\delta, r)$, it is sufficient to verify (2.14) when z runs a set of the kind $D(\delta, r)$.

If ε is a positive number, then there exists $A = A(\varepsilon) > 1$ such that

$$2^{\operatorname{Re}\alpha-1/2}\int_A^\infty u^{\operatorname{Re}\alpha-1/2}\exp(-\delta u)\,du < \varepsilon.$$

Using the representation

$$\Gamma(\alpha+1/2)(-z)^{-\alpha/2-1/4} = \int_0^\infty u^{\alpha-1/2}\exp[-(-z)^{1/2}u]\,du,\ \ z\in\mathbb{C}\setminus[0,\infty),$$

we can write

$$\begin{aligned} E_n^{(\alpha)}(z) &= 2^{\alpha-1/2}\Gamma(\alpha+1/2)(-z)^{-\alpha/2-1/4} - T_n^{(\alpha)}(z) \\ &= \int_0^A u^{\alpha-1/2}\{2^{\alpha-1/2} - R_n^{(\alpha)}(z,u)\}\exp[-(-z)^{1/2}u]\,du \\ &\quad - \int_A^{(4n+1)^{1/2}} u^{\alpha-1/2}R_n^{(\alpha)}(z,u)\exp[-(-z)^{1/2}u]\,du \\ &\quad + 2^{\alpha-1/2}\int_A^\infty u^{\alpha-1/2}\exp[-(-z)^{1/2}u]\,du \end{aligned}$$

provided $n > n_0(A) = [(A^2-1)/4]$.

The inequality $|R_n^{(\alpha)}(z,u)\exp[-(-z)^{1/2}u]| \leq 2^{\operatorname{Re}\alpha-1/2}\exp(3r/2)\exp(-\delta u)$ holds for $u \in [0,(4n+1)^{1/2}]$ and $z \in D(\delta, r)$. Hence, for $z\in D(\delta,r)$ the estimate

$$\begin{aligned} &\left|\int_A^{(4n+1)^{1/2}} u^{\alpha-1/2}R_n^{(\alpha)}(z,u)\exp[-(-z)^{1/2}u]\,du\right| \\ &\leq 2^{\operatorname{Re}\alpha-1/2}\exp(3r/2)\int_A^{(4n+1)^{1/2}} u^{\operatorname{Re}\alpha-1/2}\exp(-\delta u)\,du \\ &< 2^{\operatorname{Re}\alpha-1/2}\exp(3r/2)\int_A^\infty u^{\operatorname{Re}\alpha-1/2}\exp(-\delta u)\,du < \varepsilon\exp(3r/2). \end{aligned}$$

holds.

Since $\lim_{n\to\infty} R_n^{(\alpha)}(z,u) = 2^{\alpha-1/2}$ uniformly when $(z,u) \in D(\delta,r) \times [0,A]$, there exists $\nu_0 > n_0(A)$ such that the inequality $|2^{\alpha-1/2} - R_n^{(\alpha)}(z,u)| < \varepsilon$ holds for $z \in D(\delta,r), u \in [0,A]$ and $n > \nu_0$. Hence, for $z \in D(\delta,r)$ and $n > \nu_0$,

$$|E_n^{(\alpha)}(z)| \le \varepsilon \int_0^A u^{\operatorname{Re}\alpha-1/2}\exp(-\delta u) + \varepsilon\exp(3r/2) + \varepsilon$$

$$< \{\Gamma(\operatorname{Re}\alpha+1/2)\delta^{-\operatorname{Re}\alpha-1/2} + \exp(3r/2)+1\}\varepsilon.$$

As a result of the previous considerations we obtain that if $z \in \mathbb{C}\setminus[0,\infty)$, then

$$K_{n,1}^{(\alpha)}(z) = K(\alpha,z)(4n+1)^{-\alpha/2-1/4}\{1+k_{n,1}^{(\alpha)}(z)\},$$

where $K(\alpha,z) = 2^{\alpha-1/2}\Gamma(\alpha+1/2)(-z)^{-\alpha/2-1/4}\exp(z/2)$ and

$$\lim_{n\to\infty} k_{n,1}^{(\alpha)}(z) = 0 \tag{2.15}$$

uniformly on every compact subset of the region $\mathbb{C}\setminus[0,\infty)$.

Further, if $z \in D(\delta,r)$ and $n = 1,2,3,\dots$, then we find easily that

$$|K_{n,2}^{(\alpha)}(z)| \le \exp(r/2)\exp[-(4n+1)^{1/2}\delta]\int_0^1 (1-t^2)^{\operatorname{Re}\alpha-1/2}\,dt. \tag{2.16}$$

For sufficiently large n the inequality $|\tilde{h}_n(z,t)| < \varepsilon$ for $z \in D(\delta,r)$ and $t \in [0,1]$ implies the estimate

$$|K_{n,3}^{(\alpha)}(z)| \le \varepsilon\int_0^1 (1-t^2)^{\operatorname{Re}\alpha-1/2}\exp[-(4n+1)^{1/2}\delta(1-t)]\,dt.$$

Setting $1-t = (4n+1)^{-1/2}u$, we obtain that

$$|(4n+1)^{\alpha/2+1/4}K_{n,3}^{(\alpha)}(z)| \le \varepsilon\exp(r/2)\int_0^\infty u^{\operatorname{Re}\alpha-1/2}\exp(-\delta u)\,du \tag{2.17}$$

provided that $z \in D(\delta,r)$ and n is large enough.

Let us define $k_{n,j}^{(\alpha)}(z) = \{K(\alpha,z)\}^{-1}(4n+1)^{\alpha/2+1/4}K_{n,j}^{(\alpha)}(z),\ j=2,3$ and

$$\tilde{k}_n^{(\alpha)}(z) = \sum_{j=1}^{3} k_{n,j}^{(\alpha)}(z),\ z\in\mathbb{C}\setminus[0,\infty),\ n=1,2,3,\dots. \tag{2.18}$$

Then from (2.13) we obtain that for $z \in \mathbb{C}\setminus[0,\infty)$ and $n = 1,2,3,\dots$

$$\sqrt{\pi}2^{-\alpha+1/2}(-z)^{\alpha/2+1/4}\exp(-z/2)L_n^{(\alpha)}(z)$$

$$= \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}(4n+1)^{-\alpha/2-1/4}\exp[(4n+1)^{1/2}(-z)^{1/2}]\{1+\tilde{k}_n^{(\alpha)}(z)\}.$$

Moreover, from (2.15), (2.16), (2.17) and (2.18) it follows that $\lim_{n\to\infty} \tilde{k}_n^{(\alpha)}(z) = 0$ uniformly on every compact subset of the region $\mathbb{C} \setminus [0, \infty)$.

Since $\Gamma(n+\alpha+1)/\Gamma(n+1) = n^\alpha(1+O(n^{-1}))$ [Appendix, (1.13)],

$$(4n+1)^{-\alpha/2-1/4} = 2^{-\alpha-1/2}n^{-\alpha/2-1/4}(1+O(n^{-1}))$$

and

$$\exp[(4n+1)^{1/2}(-z)^{1/2} - 2(-z)^{1/2}\sqrt{n}] = 1 + O(n^{-1/2})$$

uniformly on every compact subset of the region $\mathbb{C}\setminus[0,\infty)$. Then we conclude that the asymptotic formula (2.5) holds in this region when $\operatorname{Re}\alpha \geq 1/2$. Moreover, under this condition the equality (2.6) holds uniformly on every compact subset of the region $\mathbb{C} \setminus [0, \infty)$. Finally, by means of the relation [Chapter I, Exercise 6, (a)],

$$L_n^{(\alpha)}(z) = L_n^{(\alpha+1)}(z) - L_{n-1}^{(\alpha+1)}(z),\ n = 1, 2, 3, \ldots \tag{2.19}$$

we extend the validity of (2.11) for arbitrary $\alpha \in \mathbb{C}$.

The asymptotics of Laguerre polynomials on the real and positive semiaxis is given by the following proposition:

(III.2.5) *If* $\alpha \in \mathbb{R}, x \in (0, \infty)$ *and* $n = 1, 2, 3, \ldots$ *then*

$$\sqrt{\pi}x^{\alpha/2+1/4}\exp(-x/2)L_n^{(\alpha)}(x) \tag{2.20}$$

$$= n^{\alpha/2-1/4}\{\cos[2\sqrt{nx} - \alpha\pi/2 - \pi/4] + \theta_n^{(\alpha)}(x)n^{-1/2}\}.$$

The sequence $\{\theta_n^{(\alpha)}(x)\}_{n=1}^\infty$ *is uniformly bounded on every segment* $[\varepsilon, \omega]$ *such that* $0 < \varepsilon < \omega < \infty$ *and, hence, on each compact subset of the semiaxis* $(0, \infty)$.

3. Asymptotic formulas for Laguerre and Hermite associated functions

3.1 The asymptotics of the Laguerre associated functions $\{M_n^{(\alpha)}(z)\}_{n=0}^\infty$ when n tends to infinity and $|z|$ is bounded is in some sence reciprocal to that of the Laguerre polynomials in the region $\mathbb{C}\setminus[0,\infty)$. More precisely, the following proposition is true:

(III.3.1) *If* $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$ *then for* $z \in \mathbb{C} \setminus [0, \infty)$ and $n = 1, 2, 3, \ldots$

$$-\sqrt{\pi}(-z)^{-\alpha/2+1/4}\exp(z/2)M_n^{(\alpha)}(z) \tag{3.1}$$

$$= n^{\alpha/2-1/4}\exp[-2(-z)^{1/2}\sqrt{n}]\{1 + \mu_n^{(\alpha)}(z)\}.$$

Moreover, the sequence of functions $\{n^{1/2}\mu_n^\alpha(z)\}_{n=1}^\infty$, *holomorphic in the region* $\mathbb{C} \setminus [0, \infty)$, *is uniformly bounded on every compact subset of this region.*

If $\alpha > -1$, then the asymptotic formula (3.1) follows from the representation [Chapter II, (4.6)] of the Laguerre associated functions by means of Tricomi's degenerate hypergeometric functions, and the asymptotic formula [Appendix, (3.16)] for these functions.

By means of the relation $nM_n^{(\alpha)}(z) = (M_{n-1}^{(\alpha+1)}(z))'$, $n = 1, 2, 3, \ldots$ [Chapter I, Exercise 25] one can prove that if a representation of the kind (3.1) holds for some $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$ in the region $\mathbb{C} \setminus [0, \infty)$, then the same is true for the functions $\{M_n^{(\alpha-1)}(z)\}_{n=1}^{\infty}$. The last assertion follows from the well-known fact that if a set of complex-valued functions, holomorphic in a domain $G \subset \mathbb{C}$, is uniformly bounded on every compact subset of G, then the same is true for the set of their derivatives.

Let α be an arbitrary complex number and define for

$$\mu_n^{(\alpha)}(z) = -\sqrt{\pi}(-z)^{-\alpha/2+1/4} \exp(z/2) n^{-\alpha/2+1/4} \exp[2(-z)^{1/2}\sqrt{n}] M_n^{(\alpha)}(z) - 1.$$

for $z \in \mathbb{C} \setminus [0, \infty)$ and $n = 1, 2, 3, \ldots$. Thus, (3.1) holds for each $\alpha \in \mathbb{C}$ and each $n = 1, 2, 3, \ldots$ provided $z \in \mathbb{C} \setminus [0, \infty)$. Moreover, we shall prove that:

(III.3.2) *If $\alpha \in \mathbb{C}$ is fixed, then*

$$\mu_n^{(\alpha)}(z) = O(n^{-1/4}) \tag{3.2}$$

uniformly on every compact subset of $\mathbb{C} \setminus [0, \infty)$ when $n \to \infty$.

Proof. It is based on the integral representation [Chapter II, (2.18)], i.e.

$$M_n^{(\alpha)}(z) = -\int_0^{\infty} \frac{t^{n+\alpha} \exp(-t)}{(t-z)^{n+1}}\, dt,$$

which holds for $z \in \mathbb{C} \setminus [0, \infty)$ and $n = 0, 1, 2, \ldots$ provided $\operatorname{Re}\alpha > -1$. Further we suppose that $\operatorname{Re}\alpha \geq 1$.

Let K be an arbitrary compact subset of the region $\mathbb{C} \setminus [0, \infty)$. Then, there exists a positive integer $n_0 = n_0(K, \alpha)$ such that $(n+\alpha)(-z)+(z+\alpha-1)^2/4 \in \mathbb{C} \setminus (-\infty, 0]$ for $z \in K$ and $n \geq n_0$. Moreover,

$$\tau_n^{(\alpha)}(z) = \sqrt{(n+\alpha)(-z) + (z+\alpha-1)^2/4} + (z+\alpha-1)/2$$

is that root of the equation

$$\frac{\partial}{\partial \zeta}\left\{\frac{\zeta^{n+\alpha} \exp(-\zeta)}{(\zeta - z)^{n+1}}\right\} = 0, \tag{3.3}$$

for which $\operatorname{Re}\tau_n^{(\alpha)}(z) > 0$ for $z \in K$ and $n \geq n_0$. Further,

$$\tau_n^{(\alpha)}(z) = (-z)^{1/2} n^{1/2} + (z+\alpha-1)/2 + O(n^{-1/2}), \tag{3.4}$$

and

$$\sigma_n^{(\alpha)}(z) = -z(\tau_n^{(\alpha)}(z))^{-1} = (-z)^{1/2}n^{-1/2} - (z+\alpha-1)/2n + O(n^{-3/2}) \tag{3.5}$$

uniformly on K when n tends to infinity.

Suppose that $n \geq n_0$ and denote by $r_n^{(\alpha)}(z)$ the ray starting at the origin and passing trough the point $\tau_n^{(\alpha)}(z)$. The function $\zeta^{n+\alpha}\exp(-\zeta)(\zeta - z)^{-n-1}$ is holomorphic in the angular domain bounded by the rays $[0,\infty)$ and $r_n^{(\alpha)}(z)$ and it is continuous on its closure. Then the Cauchy integral theorem yields

$$M_n^{(\alpha)}(z) = -\int_{r_n^{(\alpha)}(z)} \frac{\zeta^{n+\alpha}\exp(-\zeta)}{(\zeta - z)^{n+1}}\,d\zeta.$$

From this representation we obtain

$$M_n^{(\alpha)}(z) = -(\tau_n^{(\alpha)}(z))^{\alpha}\int_0^{\infty} \frac{t^{n+\alpha}\exp(-\tau_n^{(\alpha)}(z)t)}{(t+\sigma_n^{(\alpha)}(z))^{n+1}}\,dt.$$

Then, substituting $1 + n^{-1/4}t$ for t, we find that

$$M_n^{(\alpha)}(z) = -E_n^{(\alpha)}(z)\Big\{S_n^{(\alpha)}(z) + R_n^{(\alpha)}(z)\Big\}, \tag{3.6}$$

where

$$E_n^{(\alpha)}(z) = \frac{n^{-1/4}(\tau_n^{(\alpha)}(z))^{\alpha}\exp(-\tau_n^{(\alpha)}(z))}{(1+\sigma_n^{(\alpha)}(z))^{n+1}}, \tag{3.7}$$

$$S_n^{(\alpha)}(z) = \int_{-n^{1/4}}^{n^{1/4}} F_n^{(\alpha)}(z; n^{-1/4}t)\,dt,$$

$$R_n^{(\alpha)}(z) = \int_{n^{1/4}}^{\infty} F_n^{(\alpha)}(z; n^{-1/4}t)\,dt, \tag{3.8}$$

and

$$F_n^{(\alpha)}(z;u) = \frac{(1+u)^{n+\alpha}\exp(-\tau_n^{(\alpha)}(z)u)}{(1+(1+\sigma_n^{(\alpha)}(z))^{-1}u)^{n+1}}, \quad -1 < u < \infty.$$

Denoting

$$\begin{aligned}\omega_n(\alpha, z; u) = (n+\alpha)\log(1+u) - \tau_n^{(\alpha)}(z)u \\ -(n+1)\log(1+(1+\sigma_n^{(\alpha)}(z))^{-1}u),\end{aligned}$$

we can write

$$F_n^{(\alpha)}(z;u) = \exp(\omega_n(\alpha, z; u)). \tag{3.9}$$

Further, since $\tau_n^{(\alpha)}(z)$ is a root of the equation (3.3), $\omega_n(\alpha, z; 0) = \omega_n'(\alpha, z; 0) = 0$ and, hence,

$$\omega_n(\alpha, z; u) = (1/2)\omega_n''(\alpha, z; 0)u^2 + \omega_n^*(\alpha, z; u), \tag{3.10}$$

where

$$\omega_n^*(\alpha, z; u) = \frac{u^3}{2}\int_0^1 (1-s)^2 \omega_n'''(\alpha, z; us)\, ds. \tag{3.11}$$

We assert that if n_0 is large enough, then

$$\mathrm{Re}\{\omega_n'''(\alpha, z; u)\} > 0, \ -1 < u < \infty, \tag{3.12}$$

for $z \in K$ and $n > n_0$. Indeed,

$$\omega_n'''(\alpha, z; u) = 2(n+\alpha)(1+u)^{-3}$$

$$\begin{aligned} &-2(n+1)(1+\sigma_n^{(\alpha)}(z))^{-3}(1+(1+\sigma_n^{(\alpha)}(z))^{-1}u)^{-3} \qquad (3.13)\\ &= 2(n+\alpha)(1+u)^{-3} - 2(n+1)(1+u+\sigma_n^{(\alpha)}(z))^{-3}\\ &= 2(\alpha-1)(1+u)^{-3} + 2(n+1)(1+u)^{-3}\left\{1 - \left(\frac{1+u}{1+u+\sigma_n^{(\alpha)}(z)}\right)^3\right\}. \end{aligned}$$

Further, from (3.5) it follows $\mathrm{Re}(\sigma_n^{(\alpha)}(z)) > 0$ for $z \in K$ and $n > n_0$. Hence, $|(1+u)(1+u+\sigma_n^{(\alpha)}(z))^{-1}| < 1$, and, therefore, $\mathrm{Re}\{(1+u)^3(1+u+\sigma_n^{(\alpha)}(z))^{-3}\} < 1$.

We have

$$S_n^{(\alpha)}(z) = \sum_{j=1}^{3} S_{n,j}^{(\alpha)}(z),$$

where

$$S_{n,1}^{(\alpha)}(z) = \int_{-n^{1/12}}^{n^{1/12}} \exp((1/2)\omega_n''(\alpha, z; 0)n^{-1/2}t^2)\, dt,$$

$$\begin{aligned} &S_{n,2}^{(\alpha)}(z) \qquad (3.14)\\ &= \int_{-n^{1/12}}^{n^{1/12}} \exp((1/2)\omega_n''(\alpha, z; 0)n^{-1/2}t^2)(\exp(\omega_n^*(\alpha, z; n^{-1/4}t) - 1)\, dt, \end{aligned}$$

and

$$S_{n,3}^{(\alpha)}(z) = \int_{n^{1/4} \le |t| \le n^{1/12}} F_n^{(\alpha)}(z; n^{-1/4}t)\, dt.$$

Now we are to prove that

$$F_n^{(\alpha)}(z; u) = O(\exp(-(-z)^{1/2}n^{1/2}u^2/2)) \tag{3.15}$$

uniformly for $z \in K, -1 < u < 1$ and $n > n_0$. First observe that from (3.5) it follows

$$\omega_n''(\alpha, z; 0) = -n - \alpha + (n+1)(1 + \sigma_n^{(\alpha)}(z))^{-2} \tag{3.16}$$

$$= -2(-z)^{1/2} n^{1/2} + \eta_n^{(\alpha)}(z),$$

where $\eta_n^{(\alpha)}(z) = O(1)$ uniformly for $z \in K$ and $n > n_0$.

Suppose that $-1 < u \leq 0$. Then (3.9), (3.10), (3.11), (3.12) and (3.13) yield

$$F_n^{(\alpha)}(z; u) = O(\exp((1/2)\omega_n(\alpha, z; 0)u^2)) = O(\exp(-(-z)^{1/2} n^{1/2} u^2))$$

uniformly for $z \in K, u \in (-1, 0]$ and $n > n_0$.

From the equality

$$F_n^{(\alpha)}(z) = \frac{(1+u)^{\alpha-1} \exp(-\tau_n^{(\alpha)}(z)u)}{(1 - \sigma_n^{(\alpha)}(z)(1 + \sigma_n^{(\alpha)}(z))^{-1} u (1+u)^{-1})^{n+1}}$$

we obtain

$$\log F_n^{(\alpha)}(z; u) = (\alpha - 1)\log(1+u) - \tau_n^{(\alpha)}(z)u$$

$$+ (n+1)\log(1 - \sigma_n^{(\alpha)}(z)(1 + \sigma_n^{(\alpha)}(z))^{-1} u (1+u)^{-1})$$

$$= (\alpha - 1)\log(1+u) - \tau_n^{(\alpha)}(z)u + (n+1)\sigma_n^{(\alpha)}(z)(1 + \sigma_n^{(\alpha)}(z))^{-1} u (1+u)^{-1} + O(1),$$

and then the asymptotic formulas (3.4) and (3.5) yield

$$\log F_n^{(\alpha)}(z; u) = -(-z)^{1/2} n^{1/2} u + (-z)^{1/2} n^{1/2} u (1+u)^{-1} + O(1)$$

$$= -(-z)^{1/2} n^{1/2} u^2 (1+u)^{-1} + O(1) = O(-(-z)^{1/2} n^{1/2} u^2/2)$$

uniformly for $z \in K, u \in [0, 1)$ and $n > n_0$.

Since $K \subset \mathbb{C} \setminus [0, \infty)$ is a compact set, there exists a positive δ such that $\mathrm{Re}(-z)^{1/2} \geq \delta$ for $z \in K$. Then from (3.15) we obtain

$$S_{n,3}^{(\alpha)}(z) = O\left(\int_{n^{1/4}}^{n^{1/12}} \exp(-(\delta/2)t^2)\, dt\right)$$

$$= O\left(\int_{n^{1/2}}^{\infty} \exp(-(\delta/2)t)\, dt\right) = O\left(\exp(-(\delta/2)n^{1/2})\right).$$

Further, from (3.11) and (3.13) we obtain that $|\omega_n^*(\alpha, z; n^{-1/4}t)| = O(n^{-1/4}|t|^3)$ and, hence, $|\exp(\omega_n^*(\alpha, z; n^{-1/4})) - 1| \leq |\omega_n^*(\alpha, z; n^{-1/4}t)| \exp(|\omega_n^*(\alpha, z; n^{-1/4})|) = O(n^{-1/4}|t|^3)$ provided $|t| \leq n^{1/12}$ and $z \in K$. Then, (3.14) yields

$$S_{n,2}^{(\alpha)}(z) = O\left(n^{-1/4} \int_0^{\infty} \exp(-\delta t^2) t^3\, dt\right) = O(n^{-1/4})$$

uniformly for $z \in K$ and $n > n_0$.

Further,

$$S_{n,1}^{(\alpha)}(z) = 2\int_0^{\infty} \exp((1/2)\omega_n''(\alpha, z; 0)n^{-1/2}t^2)\, dt \tag{3.17}$$
$$-2\int_{n^{1/12}}^{\infty} \exp((1/2)\omega_n''(\alpha, z; 0)n^{-1/2}t^2)\, dt.$$

Cauchy's theorem enable us to replace the integration on the real and non-negative semiaxes by the integration on the ray starting at the origin and passing trough the point $-(1/2)\omega_n''(\alpha, z; 0)n^{-1/2}$ and, thus, to obtain that

$$2\int_0^{\infty} \exp((1/2)\omega_n''(\alpha, z; 0)n^{-1/2}t^2)\, dt$$
$$= \int_0^{\infty} t^{-1/2} \exp((1/2)\omega_n''(\alpha, z; 0)n^{-1/2}t)\, dt$$
$$= ((1/2)\omega_n''(\alpha, z; 0)n^{-1/2})^{-1/2} \int_0^{\infty} t^{-1/2} \exp(-t)\, dt$$
$$= \sqrt{\pi}((-z)^{1/2} + O(n^{-1/2}))^{-1/2} = \sqrt{\pi}(-z)^{-1/4}(1 + O(n^{-1/2}))$$

uniformly for $z \in K$ when n tends to infinity.

Denoting $\lambda = \sup_{z\in K, n>n_0} |\eta_n^{(\alpha)}(z)|$, and using (3.16), for the second integral in (3.17) we obtain

$$\int_{n^{1/12}}^{\infty} \exp((1/2)\omega_n''(\alpha, z; 0)n^{-1/2}t^2)\, dt = O\biggl(\int_{n^{1/12}}^{\infty} \exp(-(\delta - \lambda n^{-1/2})t^2)\, dt\biggr)$$
$$= O\biggl(\int_{n^{1/6}}^{\infty} \exp(-(\delta - \lambda n^{-1/2})t)\, dt\biggr) = O(\exp(-(\delta - \lambda n^{-1/2})n^{1/6}))$$
$$= O(\exp(-\delta n^{1/6}))$$

uniformly for $z \in K$ and $n > n_0$.

It remains to estimate the integral (3.8). We have

$$R_n^{(\alpha)}(z) = n^{1/4} \exp(\tau_n^{(\alpha)}(z))(1 + \sigma_n^{(\alpha)}(z))W_n^{(\alpha)}(z), \tag{3.18}$$

where

$$W_n^{(\alpha)}(z) = \int_2^{\infty} \frac{t^{n+\alpha}\exp(-\tau_n^{(\alpha)}(z)t)}{(t + \sigma_n^{(\alpha)}(z))^{n+1}}\, dt.$$

Since $\operatorname{Re}\sigma_n^{(\alpha)}(z) > 0$ for $z \in K$ and $n > n_0$,

$$|W_n^{(\alpha)}(z)| \le \int_2^{\infty} t^{\operatorname{Re}\alpha - 1} \exp(-\xi_n^{(\alpha)}(z)t)\, dt,$$

where $\xi_n^{(\alpha)}(z) = \mathrm{Re}\{\tau_n^{(\alpha)}(z)\}$. Substituting $(\xi_n^{(\alpha)}(z))^{-1}t$ for t, we obtain

$$|W_n^{(\alpha)}(z)| \le (\xi_n^{(\alpha)}(z))^{-\mathrm{Re}\,\alpha} \int_{2\xi_n^{(\alpha)}(z)}^{\infty} t^{\mathrm{Re}\,\alpha-1} \exp(-t)\, dt. \tag{3.19}$$

If $k > \mathrm{Re}\,\alpha$ is a positive integer, then integrating by parts we obtain

$$\int_{2\xi_n^{(\alpha)}(z)}^{\infty} t^{\mathrm{Re}\,\alpha-1} \exp(-t)\, dt = \exp(-2\xi_n^{(\alpha)}(z)) \sum_{s=1}^{k} C_s^{(\alpha)} (\xi_n^{(\alpha)}(z))^{\mathrm{Re}\,\alpha-s}$$
$$+ \int_{2\xi_n^{(\alpha)}(z)}^{\infty} t^{\mathrm{Re}\,\alpha-k} \exp(-t)\, dt,$$

where $C_s^{(\alpha)}$, $s = 1, 2, 3, \ldots, k$, are constants not depending on n and z.

From the above equality it follows that

$$\int_{2\xi_n^{(\alpha)}(z)}^{\infty} t^{\mathrm{Re}\,\alpha-1} \exp(-t)\, dt = O\Big((\xi_n^{(\alpha)}(z))^{\mathrm{Re}\,\alpha-1} \exp(-2\xi_n^{(\alpha)}(z))\Big) \tag{3.20}$$

uniformly for $z \in K$ and $n > n_0$.

Since $\inf_{z\in K, n\ge n_0} n^{-1/2}\xi_n^{(\alpha)}(z) > 0$, (3.19) and (3.20) yield that

$$|W_n^{(\alpha)}(z)| = O(n^{-1/2} \exp(-2(-z)^{1/2} n^{1/2}))$$

uniformly for $z \in K$ and $n > n_0$.

Since $\exp(\tau_n^{(\alpha)}(z))(1+\sigma_n^{(\alpha)}(z))^{n+1} = O(\exp(2(-z)^{1/2}n^{1/2}))$ uniformly for $z \in K$ and $n > n_0$, from (3.18) we obtain that $R_n^{(\alpha)}(z) = O(n^{-1/4})$. Finally, from (3.4), (3.5) and (3.7) it follows that

$$E_n^{(\alpha)}(z) = \exp(-z/2)(-z)^{\alpha/2} n^{\alpha/2-1/4} \exp(-2(-z)^{1/2}\sqrt{n})(1 + O(n^{-1/2})),$$

and the validity of (3.2), under the assumption $\mathrm{Re}\,\alpha \ge 1$, follows from (3.6). Again the relation [Chapter I, Exercise 25] enables us to verify (3.2) for arbitrary $\alpha \in \mathbb{C}$.

Using [Chapter II, (4.6)] and the representation [Appendix, (3.16)] we can state the following proposition:

(III.3.3). *If $\alpha \in \mathbb{C} \setminus \mathbb{Z}^-$ and $n = 0, 1, 2, \ldots$ are fixed, then*

$$M_n^{(\alpha)}(z) = (-1)^n \Gamma(n + \alpha + 1) z^{-n-1} \{1 + O(z^{-1})\} \tag{3.21}$$

when z tends to infinity in the region $\mathbb{C} \setminus [0, \infty)$.

3.2 By means of the relations [Chapter I, (5.26),(5.27)] and the representations (3.1), (3.2) one can easily find asymptotic formulas for the Hermite associated functions $\{G_n^{\pm}(z)\}_{n=0}^{\infty}$. We give here only the statements of the corresponding propositions and leave their proofs to the reader.

(III.3.4): (a) *The representation*

$$(\pi\sqrt{2})^{-1}\exp(z^2/2)G_n^+(z) \tag{3.22}$$

$$= (-i)^{n+1}(2n/e)^{n/2}\exp(iz\sqrt{2n+1})\{1+k_n^+(z)\},\ n=1,2,3,\dots$$

holds in the half-plane $H^+ : \operatorname{Im} z > 0$, where the complex functions $\{k_n^+(z)\}_{n=1}^{\infty}$ are holomorphic in H^+ and $\lim_{n\to\infty} k_n^+(z) = 0$ uniformly on every compact subset of H^+.

(b) *The representation*

$$(\pi\sqrt{2})^{-1}\exp(z^2/2)G_n^-(z)$$

$$= i^{n+1}(2n/e)\exp(-iz\sqrt{2n+1})\{1+k_n^-(z)\},\ n=1,2,3,\dots \tag{3.23}$$

holds in the region $H^- : \operatorname{Im} z < 0$, where $\{k_n^-(z)\}_{n=0}^{\infty} = \{\overline{k_n^+(\overline{z})}\}_{n=1}^{\infty}$ are holomorphic functions in H^-, and $\lim_{n\to\infty} k_n^-(z) = 0$ uniformly on every compact subset of H^-.

(III.3.5). *If $n = 0,1,2,\dots$ is fixed and z approaches infinity in $\mathbb{C}\setminus\mathbb{R}$, then*

$$G_n^{\pm}(z) = \sqrt{\pi}n!z^{-n-1}\{1+O(z^{-2})\}. \tag{3.24}$$

4. Inequalities for Laguerre and Hermite polynomials

4.1 Throughout this book we do not use the asymptotics of the Laguerre polynomials on the real line. But now we shall point out the special importance of some inequalities for these polynomials on the ray $[0,\infty)$ as well as on some (closed) subregions of the complex plane.

(III.4.1) (a) *If $\alpha \in \mathbb{R}\setminus\mathbb{Z}^-$, $0 < \omega < \infty$ and $n \to \infty$, then*

$$L_n^{(\alpha)}(x) = O(n^a),\ a = \max(\alpha/2 - 1/4, \alpha) \tag{4.1}$$

uniformly with respect to $x \in [0,\omega]$;

(b)*If $1 \le \omega < \infty$ and $n \to \infty$, then*

$$L_n^{(\alpha)}(x) = O(\exp(x/2)x^{-(\alpha+1)/2}n^{\alpha/2+1/6}) \tag{4.2}$$

uniformly with respect to $x \in [\omega,\infty)$.

The inequality (4.2) reflects the behaviour of $L_n^{(\alpha)}(x)$ as a function of n and the real variable x when $n\to\infty$ and $x\to\infty$. An analogous inequality holds for $L_n^{(\alpha)}(z)$ as a function of n and the complex variable z when $n\to\infty$ and $z\to\infty$ in suitable domains of the complex plane. In order to state the corresponding proposition we recall that for $\lambda \in (0,\infty)$, by $p(\lambda)$ we denote the parabola with

Cartesian equation $y^2 = 4\lambda^2(x+\lambda^2)$. The same parabola can be described by the equation $\mathrm{Re}(-z)^{1/2} = \lambda$. Let us denote by $\Delta(\lambda)$ the interior of $p(\lambda)$, i.e. $\Delta(\lambda) = \{z \in \mathbb{C} : \mathrm{Re}(-z)^{1/2} < \lambda\}$.

(III.4.2) *If* $\lambda \in (0,\infty), \rho > \max(1, 2\lambda^2)$ *and* $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$, *then there exists a constant* $A = A(\lambda,\rho,\alpha)$ *such that for* $z = x+iy \in \tilde{\Delta}(\lambda,\rho) = \overline{\Delta(\lambda)} \cap \{z \in \mathbb{C} : |z| \geq \rho\}$ *and* $n = 1,2,3,\ldots$

$$|L_n^{(\alpha)}(z)| \leq A|z|^{-\alpha/2-1/4} n^{\alpha/2-1/4} \exp(x + 2\lambda\sqrt{n}). \tag{4.3}$$

Proof. We have to prove that the sequence

$$A_n^{(\alpha)}(z) = z^{\alpha/2+1/4} n^{-\alpha/1+1/4} \exp(-z-2\lambda\sqrt{n}) L_n^{(\alpha)}(z),\ n = 1,2,3\ldots,$$

defined in the region $\mathbb{C} \setminus (-\infty, 0]$, is uniformly bounded on the closed set $\tilde{\Delta}(\lambda,\rho)$.

If we take into account the integral representation [Chapter II, (2.13)] of the Laguerre polynomials as well as the definition of the function B_α by [Chapter II, (2.5)], then we have

$$A_n^{(\alpha)}(z) = (n!)^{-1} z^{1/4} n^{-\alpha/2+1/4} \exp(-2\lambda\sqrt{n}) \int_0^\infty t^{n+\alpha/2} J_\alpha(2\sqrt{zt})\,dt$$

for $z \in \mathbb{C} \setminus (-\infty, 0]$.

Remark. The last representation holds when $n+\alpha > -1$. Further we suppose that $n+\alpha > 1$.

For $z = R\exp i\theta$ with $-\pi/2 < \theta < \pi/2$ we denote

$$A_{n,1}^{(\alpha)}(z) = (n!)^{-1} z^{1/4} n^{-\alpha/2+1/4} \exp(-2\lambda\sqrt{n}) \int_0^{1/R} t^{n+\alpha/2} \exp(-t) J_\alpha(2\sqrt{zt})\,dt$$

and

$$A_{n,2}^{(\alpha)}(z) = (n!)^{-1} z^{1/4} n^{-\alpha/2+1/4} \exp(-2\lambda\sqrt{n}) \int_{1/R}^\infty t^{n+\alpha/2} \exp(-t) J_\alpha(2\sqrt{zt})\,dt.$$

Since $A_n^{(\alpha)}(z) = A_{n,1}^{(\alpha)}(z) + A_{n,2}^{(\alpha)}(z)$ when $z \in \mathbb{C} \setminus (-\infty, 0]$ and $n = 1,2,3,\ldots$, we are to prove that every of the sequences $\{A_{n,1}^{(\alpha)}(z)\}_{n=1}^\infty$ and $\{A_{n,2}^{(\alpha)}(z)\}_{n=1}^\infty$ is uniformly bounded on the set $\tilde{\Delta}(\lambda,\rho)$.

The power series expansion of $z^{-\alpha/2} J_\alpha(z)$ [Appendix, (2.2)] yields

$$|J_\alpha(2\sqrt{zt})| \leq \sum_{k=0}^\infty \frac{t^{k+\alpha/2} R^{k+\alpha/2}}{k!\Gamma(k+\alpha+1)} \leq C_\alpha R^{\alpha/2} t^{\alpha/2},\ t \in (0, R^{-1}),$$

where

$$C_\alpha = \sum_{k=0}^{\infty} \{k!\Gamma(k+\alpha+1)\}^{-1}.$$

Since $R \geq \rho \geq 1$ and $n+\alpha > 1$, the function $t^{n+\alpha}\exp(-t)$ is increasing on the segment $[0, R^{-1}]$ and, hence,

$$\int_0^{1/R} t^{n+\alpha}\exp(-t)\,dt < R^{-n-\alpha-1}\exp(-1/R) < R^{-n-\alpha-1}.$$

Then for $z \in \tilde{\Delta}(\lambda,\rho)$ and $n+\alpha > 1$ we obtain that

$$|A_{n,1}^{(\alpha)}(z)| \leq C_\alpha (n!)^{-1} R^{-n-\alpha/2-3/4} n^{-\alpha/2+1/4}\exp(-2\lambda\sqrt{n})$$
$$< C_\alpha (n!)^{-1} n^{-\alpha/2+1/4}\exp(-2\lambda\sqrt{n}).$$

Using the asymptotic formula [Appendix, (2.16)] for the Bessel functions of the first kind, we obtain that the representation

$$J_\alpha(z) = (2/\pi z)^{1/2}\{\cos(z-\alpha\pi/2-\pi/4)(1+\xi(z))$$
$$+\sin(z-\alpha\pi/2-\pi/4)(b/z+\eta(z))\} \tag{4.4}$$

holds in the region $\mathbb{C}\setminus(-\infty,0]$, where $b \neq 0$ is a constant and the complex functions $\xi(z)$ and $\eta(z)$ are holomorphic in this region. More precisely, $\xi(z) = O(z^{-2})$ and $\eta(z) = O(z^{-3})$ when z tends to infinity and $|\arg z| \leq \pi-\delta, \delta \in (0,\pi)$. Hence, these functions are bounded on the closed set defined by the inequalities $|z| \geq 1$ and $|\arg z| \leq \pi/4$.

Suppose that $z = R\exp i\theta \in \tilde{\Delta}(\lambda,\rho)$ and $1/R \leq t < \infty$. Since $|\theta| \leq \varphi$, where $\cos\varphi = 1-2\lambda^2/R$ and $|\sqrt{tz}| = \sqrt{tR} \geq 1$, the inequalities

$$|\cos(2\sqrt{tz}-\alpha\pi/2+\pi/4)| \leq \exp(2\sqrt{tR}|\sin(\theta/2)|),$$
$$|\sin(2\sqrt{tz}-\alpha\pi/2-\pi/4)| \leq \exp(2\sqrt{tR}|\sin(\theta/2)|),$$
$$|\sin(\theta/2)| \leq \sin(\varphi/2) = \sqrt{(1-\cos\varphi)/2} = \lambda/\sqrt{R}$$

hold for $z = R\exp i\theta \in \tilde{\Delta}(\lambda,\rho)$. Then from (4.4) it follows that $|J_\alpha(2\sqrt{tz})| = O(R^{-1/4}t^{-1/4}\exp(2\lambda\sqrt{t}))$ uniformly with respect to $z \in \tilde{\Delta}(\lambda,\rho)$ and $t \in [1/R,\infty)$. Hence,

$$|A_{n,2}^{(\alpha)}(z)| = O\Big((n!)^{-1} n^{-\alpha/2+1/4}\exp(-2\lambda\sqrt{n})\int_0^\infty t^{n+\alpha/2-1/4}\exp(-t+2\lambda\sqrt{t})\,dt\Big)$$

uniformly for $z \in \tilde{\Delta}(\lambda,\rho)$ and $n+\alpha > 1$. Substituting $t = u^2/2$, from the integral representation [Appendix, (4.6)] of the Weber-Hermite functions we obtain

$$|A_{n,2}^{(\alpha)}(z)|$$

$$= O\Big((2^n n!)^{-1} n^{-\alpha/2+1/4} \exp(-2\lambda\sqrt{n})\Gamma(2n+\alpha+3/2) D_{-(2n+\alpha+3/2)}(-\lambda\sqrt{2})\Big).$$

Then, at last, the asymptotic formula [Appendix, (4.7)] as well as the Stirling formula yield that $|A_{n,2}^{(\alpha)}(z)| = O(1)$ uniformly when $z \in \tilde{\Delta}(\lambda, \rho)$ and $n + \alpha > 1$.

An inequality of the kind (4.3) holds true when α is a (nonreal) complex number. We denote by $\overline{\Delta}(\lambda)$ the closed region $\overline{\Delta(\lambda)} = \{z \in \mathbb{C} : \mathrm{Re}(-z)^{1/2} \le \lambda\}$ provided $\lambda \in (0, \infty)$ and define $\overline{\Delta}(0) = [0, \infty)$.

(III.4.3) *For $\lambda \in [0, \infty)$, $k = 0, 1, 2, \ldots$ and $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > -k - 1/2$ there exists a constant $C(\alpha, \lambda, k)$ such that*

$$|L_n^{(\alpha)}(z)| \le C(\alpha, \lambda, k) n^{k+\mathrm{Re}\,\alpha} \exp(x + 2\lambda\sqrt{n}). \tag{4.5}$$

for $z = x + iy \in \overline{\Delta}(\lambda)$ and $n \ge k + 1$.

Proof. Suppose that $\mathrm{Re}\,\alpha > -1/2$. Then the definition of the function B_α by [Chapter II, (2.5)] as well as the integral representation [Appendix, (2.14)] of the Bessel function $J_\alpha(z)$ imply that for $z \in \mathbb{C}$

$$B_\alpha(z) = \frac{2}{\sqrt{\pi}\Gamma(\alpha + 1/2)} \int_0^1 (1 - t^2)^{\alpha - 1/2} \cos(2t\sqrt{z})\, dt. \tag{4.6}$$

If we define $K(\alpha) = 2(\sqrt{\pi}|\Gamma(\alpha + 1/2)|)^{-1} \int_0^1 (1 - t^2)^{\mathrm{Re}\,\alpha - 1/2}\, dt$, then from (4.6) it follows that for $z \in \mathbb{C}$,

$$|B_\alpha(z)| \le K(\alpha) \exp(2|\,\mathrm{Im}\,\sqrt{z}|). \tag{4.7}$$

Since $|\,\mathrm{Im}\,\sqrt{z}| \le \lambda$ for $z \in \overline{\Delta}(\lambda)$, from the integral representation [Chapter II, (2.13)] it follows that for each $z = x + iy \in \overline{\Delta}(\lambda)$ and $n = 0, 1, 2, \ldots$ the estimate

$$|L_n^{(\alpha)}(z)| \le \frac{\mathrm{Const}(\alpha) \exp x}{\Gamma(n+1)} \int_0^\infty t^{n+\mathrm{Re}\,\alpha} \exp(-t + 2\lambda\sqrt{t})\, dt.$$

holds.

Replacing t by $u^2/2$ and denoting $\nu(n, \alpha) = 2(n + \mathrm{Re}\,\alpha + 1)$, then the integral representation [Appendix, (4.6)] gives

$$|L_n^{(\alpha)}(z)| \le \frac{\mathrm{Const}(\alpha, \lambda)\Gamma(\nu(n, \alpha)) \exp x}{2^{n+\mathrm{Re}\,\alpha}\Gamma(n+1)} D_{-\nu(n,\alpha)}(-\lambda\sqrt{2}).$$

Further, the asymptotic formula [Appendix, (4.7)] and the Stirling formula lead to the inequality

$$|L_n^{(\alpha)}(z)| \le \mathrm{Const}(\alpha, \lambda) n^{\mathrm{Re}\,\alpha} \exp(x + 2\lambda\sqrt{n}), \tag{4.8}$$

which is valid for $z = x + iy \in \overline{\Delta}(\lambda)$ and $n = 0, 1, 2, \ldots$.

From the relation (2.14) it follows that if $k = 0, 1, 2, \ldots$, then for $n \geq k + 1$

$$L_n^{(\alpha)}(z) = \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} L_{n-\nu}^{(\alpha+k)}(z). \tag{4.9}$$

Using the above equality as well as (4.8) we come to an inequality of the kind (4.5) provided $z = x + iy \in \overline{\Delta}(\lambda)$ and $n \geq k + 1$.

4.2 The following estimate holds for the Hermite polynomials $\{H_n(x)\}_{n=0}^\infty$ as functions of n and the real variable x:

(III.4.4) *For $x \in \mathbb{R}$ and $n = 0, 1, 2, \ldots$,*

$$|H_n(x)| \leq \exp(x^2/2)(n!2^n)^{1/2}. \tag{4.10}$$

If $\tau \in \mathbb{R}^+$, then denote by $S(\tau)$ the strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < \tau\}$. Note that by $\zeta = z^2$ the domain $S(\tau)$ is mapped onto $\Delta(\lambda)$. From **(III.4.2)** and the relations [Chapter I, (5.24), (5.25)] between the Laguerre and Hermite polynomials as well as from Stirling's formula we obtain the following assertion:

(III.4.5) *For each $\tau \in \mathbb{R}^+$ there exists a constant $B = B(\tau)$ such that*

$$|H_n(z)| \leq B(2n/e)^{n/2} \exp(x^2 + \tau\sqrt{2n+1}) \tag{4.11}$$

whenever $z = x + iy \in \overline{S(\tau)}$ and $n = 0, 1, 2, \ldots$.

5. Inequalities for Laguerre and Hermite associated functions

5.1 Suppose that $\mu \in \mathbb{R}^+$ and define $\Delta^*(\mu) = \mathbb{C} \setminus \overline{\Delta(\mu)}$. Evidently, $\Delta^*(\mu)$ is the exterior of the parabola $p(\mu)$, i.e. $\Delta^*(\mu) = \{z \in \mathbb{C} : \operatorname{Re}(-z)^{1/2} > \mu\}$.

(III.5.1) *Suppose that $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$ and define $\nu(\alpha) = \max\{1, -\alpha/2 - 3/4, -\alpha -1\}$. Then there exists a positive constant $M = M(\mu, \alpha)$ with the property* that

$$|M_n^{(\alpha)}(z)| \leq M|z|^{\alpha/2-1/4} n^{\alpha/2-1/4} \exp(-2\mu\sqrt{n}). \tag{5.1}$$

for $z \in \overline{\Delta^*(\mu)}$ and $n > \nu(\alpha)$.

Proof. We use the integral representation [Chapter II, (4.7)]. Let $\rho > \max(1, 2\mu^2)$ and define for $z \in \overline{\Delta^*(\mu)} \setminus U(0;\rho)$

$$M_{n,1}^{(\alpha)}(z) = -\frac{2(-z)^{\alpha/2}}{\Gamma(n+1)} \int_0^{1/|z|} t^{n+\alpha/2} \exp(-t) K_\alpha(2\sqrt{-zt})\, dt$$

and

$$M_{n,2}^{(\alpha)}(z) = -\frac{2(-z)^{\alpha/2}}{\Gamma(n+1)} \int_{1/|z|}^{\infty} t^{n+\alpha/2} \exp(-t) K_\alpha(2\sqrt{-zt})\, dt.$$

As a special case of the asymptotic formula [Appendix, (2.18)] we have that $K_\alpha(z) = \sqrt{\pi/2z}\exp(-z)\{1+k_\alpha(z)\}$ in the region $\mathbb{C}\setminus(-\infty,0]$, where the complex function $k_\alpha(z)$ is holomorphic in this region and, moreover, $k_\alpha(z) = O(z^{-1})$ when z tends to infinity. Hence, the function $k_\alpha(z)$ is bounded on the set $\{\mathbb{C}\setminus(-\infty,0]\}\cap\{z\in\mathbb{C} : |z|\geq 1\}$. Then, applying the integral representation [Appendix, (4.6)], the asymptotic formula [Appendix, (4.7)] as well as Stirling's formula, we find that

$$|M_{n,1}^{(\alpha)}(z)| = O\biggl(\frac{|z|^{\alpha/2-1/4}}{\Gamma(n+1)}\int_0^\infty t^{n+\alpha/2-3/4}\exp(-t-2\mu\sqrt{t})\,dt\biggr)$$

$$= O\biggl(\frac{\Gamma(2n+\alpha+3/2)|z|^{\alpha/2-1/4}}{2^n\Gamma(n+1)}D_{-(2n+\alpha+3/2)}(\mu\sqrt{2})\biggr)$$

$$= O(|z|^{\alpha/2-1/4}n^{\alpha/2-1/4}\exp(-2\mu\sqrt{n}))$$

for $z\in\overline{\Delta^*(\mu)}\setminus U(0;\rho)$ and $n>\nu(\alpha)$.

If $\alpha\in\mathbb{R}\setminus\mathbb{Z}$, then from the definition of the function K_α [Appendix, (2.6)] it follows that

$$|K_\alpha(\zeta)| = O(|\zeta|^{-|\alpha|}) \tag{5.2}$$

when ζ tends to zero in the region $\mathbb{C}\setminus(-\infty,0]$.

If $\alpha = s$ is a nonnegative integer, then [Appendix, (2.7)] gives

$$|K_\alpha(\zeta)| = O(|\zeta|^s\log|\zeta|) \tag{5.3}$$

when ζ tends to zero in the region $\mathbb{C}\setminus(-\infty,0]$. Therefore, if $\alpha\in\mathbb{R}\setminus\mathbb{Z}^-$, then (5.2) yields

$$|M_{n,1}^{(\alpha)}(z)| = O\biggl(|z|^{\alpha/2-|\alpha|/2}\int_0^{1/|z|} t^{n+\alpha/2-|\alpha|/2}\exp(-t)\,dt\biggr)$$

$$= O\Bigl(\frac{1}{\Gamma(n+1)|z|^{n+1}}\Bigr) = O\Bigl(|z|^{\alpha/2-1/4}\Gamma(n+1)|z|^{n+\alpha+3/4}\Bigr) = O\Bigl(\frac{|z|^{\alpha/2-1/4}}{\Gamma(n+1)}\Bigr)$$

provided $z\in\overline{\Delta^*(\mu)}\setminus U(0;\rho)$ and $n>\nu(\alpha)$.

Since $\lim_{n\to\infty}\{\Gamma(n+1)\}^{-1}n^{\alpha/2-1/4}\exp(2\mu\sqrt{n}) = 0$, we get finally that $|M_{n,1}^{(\alpha)}(z)| = O(|z|^{\alpha/2-1/4}n^{\alpha/2-1/4}\exp(-2\mu\sqrt{n}))$ for $z\in\overline{\Delta^*(\mu)}\setminus U(0;\rho)$ and $n>\nu(\alpha)$.

Likewise, but using (5.3), we obtain that if $\alpha = s = 0,1,2,\dots$, then

$$|M_{n,2}^{(s)}(z)| = O\biggl(\frac{1}{\Gamma(n+1)}\int_0^{1/|z|} t^{n+s}\log(1/t)\exp(-t)\,dt\biggr)$$

$$= O\biggl(\frac{1}{\Gamma(n+1)}\int_0^{1/|z|} t^{n-1}\exp(-t)\,dt\biggr) = O\biggl(\frac{1}{\Gamma(n+1)|z|^n}\biggr)$$

$$= O(|z|^{s/2-1/4} n^{s/2-1/4} \exp(-2\mu\sqrt{n}))$$

provided $z \in \overline{\Delta^*(\mu)} \setminus U(0;\rho)$ and $n > \nu(s) = 1$.

So far the validity of an inequality of the kind (5.1) is obtained for the case when $z \in \overline{\Delta^*(\mu)} \setminus U(0;\rho)$ and $n > \nu(\alpha)$. From the asymptotic formula (3.1) it follows that a similar inequality holds on the compact set $\overline{\Delta^*(\mu)} \cap \overline{U(0;\rho)}$ for $n = 1, 2, 3, \ldots$. Thus, the inequality (5.1) is valid for $z \in \overline{\Delta^*(\mu)}$ and $n > \nu(\alpha)$.

5.2 If $\tau \in \mathbb{R}$, then denote by $H^+(\tau)$ and $H^-(\tau)$ the half-planes $\{z \in \mathbb{C} : \operatorname{Im} z > \tau\}$ and $\{z \in \mathbb{C} : \operatorname{Im} z < \tau\}$, respectively. In particular, $H^+(0) = H^+$ is the upper and $H^-(0) = H^-$ is the lower half-plane. If $0 \leq \tau < \infty$, then we define $S^*(\tau) = H^+(\tau) \bigcup H^-(-\tau)$, i.e. $S^*(\tau) = \{z \in \mathbb{C} : |\operatorname{Im} z| > \tau\}$.

(III.5.2) *If $\tau \in \mathbb{R}^+$, then for $z \in \overline{S^*(\tau)}$ and $n \geq 1$*

$$|G_n^{\pm}(z)| = O((2n/e)^{n/2} \exp(-\tau\sqrt{2n+1}). \tag{5.4}$$

Proof. If $z \in \overline{S^*(\tau)}$, then $z^2 \in \overline{\Delta^*(\tau)}$ and from (5.1), and the relations (3.3) and (3.4), we have that

$$|G_{2n}^{\pm}(z)| = O(2^{2n} n! n^{-1/2} \exp(-2\tau\sqrt{n})),\ n = 1, 2, 3, \ldots,$$

$$|G_{2n+1}^{\pm}(z)| = O(2^{2n} n! \exp(-2\tau\sqrt{n})),\ n = 1, 2, 3, \ldots.$$

Using these inequalities as well as Stirling's formula we come to the inequality (5.4).

Exercises

1. Prove the asymptotic formulas (3.3) and (3.4).

2. Prove the representation (3.5).

3. Give a detailed proof of the inequality (5.4).

4. Suppose that $0 < \tau_0 < \infty$ and define the entire functions

$$G_{2n}^{(1)}(z;\tau_0) = G_{2n}^{+}(z + i\tau_0) - G_{2n}^{-}(z - i\tau_0),\ n = 0, 1, 2, \ldots,$$

$$G_{2n+1}^{(1)}(z;\tau_0) = G_{2n+1}^{+}(z + i\tau_0) + G_{2n+1}^{-}(z - i\tau_0),\ n = 0, 1, 2, \ldots$$

as well as

$$G_{2n}^{(2)}(z;\tau_0) = G_{2n}^{+}(z + i\tau_0) + G_{2n}^{-}(z - i\tau_0),\ n = 0, 1, 2, \ldots,$$

$$G_{2n+1}^{(2)}(z;\tau_0) = G_{2n+1}^{+}(z + i\tau_0) - G_{2n+1}^{-}(z - i\tau_0),\ n = 0, 1, 2, \ldots.$$

Prove the validity of the following asymptotic formulas:

$$\exp[(z^2 - \tau_0^2)/2 + \tau_0\sqrt{2n+1}] G_n^{(1)}(z)$$

$$= 2(-i)^{n+1}\pi\sqrt{2}(2n/e)^{n/2}\{\cos[(\sqrt{2n+1}-\tau_0)z] + g_n^{(1)}(z)\},$$

$$\exp[(z^2-\tau_0^2)/2 + \tau_0\sqrt{2n+1}]G_n^{(2)}(z)$$

$$= 2(-i)^{n}\pi\sqrt{2}(2n/e)^{n/2}\{\sin[(\sqrt{2n+1}-\tau_0)z] + g_n^{(2)}(z)\},$$

where $\{g_n^{(k)}(z)\}_{n=0}^{\infty}$, $k = 1, 2$ are entire functions such that

$$\lim_{n\to\infty} \exp\{|\operatorname{Im} z|\sqrt{2n+1}\} g_n^{(k)}(z) = 0, \; k = 1, 2$$

uniformly on each compact subset of the strip $S(\tau_0)$.

5. Using the asymptotic formula (2.3) prove that the sequence

$$\{\exp[-|\operatorname{Im} z|(2n+1)^{1/2}]h_{n,0}(z)\}_{n=0}^{\infty},$$

where $h_{n,0}(z) = \lambda_n^{-1}\exp(-z^2/2)H_n(z)$, $n = 0, 1, 2, \ldots$, is uniformly bounded on every compact subset of the complex plane.

Remark. The result of **5** means that the asymptotic formula (2.1) is true also in the case $s = 0$ provided that the sums involved are assumed to be zero when $s = 0$.

6. Using [Chapter I, Exercise 17], prove by induction the validity of the asymptotic formula (2.1).

7. Prove that for $\rho \in (1, \infty), \alpha, \beta \in \mathbb{C}$ there exists a constant $P(\rho, \alpha, \beta)$ such that for $z \in \overline{E(\rho)}$ and $n = 1, 2, 3, \ldots$:

$$|P_n^{(\alpha,\beta)}(z)| \le P(\rho, \alpha, \beta)n^{-1/2}\rho^n.$$

8. Verify the validity of the integral representation (2.4) by proving that the complex function $\tilde{H}_n(z)$, $n = 0, 1, 2\ldots$, defined by the equality

$$\pi 2^{-n/2}(\Gamma(n+1))^{-1}\tilde{H}_n(z)$$

$$= i^{n+1}\int_{ic-\infty}^{ic+\infty} \exp(-\zeta^2/2)\zeta^{-n-1}\cosh(\sqrt{2}z\zeta - n\pi i/2)\,d\zeta,$$

satisfies the differential equation [Chapter I, (5.8)] as well as the initial conditions $\tilde{H}_n(0) = H_n(0), \tilde{H}_n'(0) = H_n'(0)$, $n = 0, 1, 2\ldots$.

Comments and references

It is not an easy task to list all the authors who have given substantial contributions to the asymptotics of the classical orthogonal polynomials and associated functions.

The first asymptotic formulas for the Legendre polynomials in the region $\mathbb{C} \setminus [-1, 1]$ have been given by P.S. Laplace and E. Heine [G. Szegö, 1,(8.21.1)],[E. Heine, 1, vol.I, p. 174]. Its extension to Jacobi polynomials is due to G. Darboux [Szegö, 1,8.21,(2),(8.21.9)],[G. Darboux, 1].

Recently H.G. Meijer [1] gave an asymptotic expansion for Jacobi's polynomials in the region $\mathbb{C} \setminus [-1, 1]$. E. Hahn [1] proposed asymptotic expansion for these polynomials in the domain $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$. Asymptotic expansions for Jacobi polynomials and associated functions in the region $\mathbb{C} \setminus [-1, 1]$ are obtained also by D. Elliot [1].

The asymptotic formula (1.7) for Jacobi polynomials in the case when α and β are arbitrary complex numbers is proved by means of Darboux's method, i.e. using the generating function [Chapter II, (1.10)] for these polynomials. A little more precise asymptotic formula is given by G. Szegö [1,(8.21.11)] provided that α and β are real. As it is pointed out [G. Szegö,1,8.4,(4)] it can be obtained also by applying Darboux's method. Therefore, it seems that [G. Szegö, 1, Theorem 8.21.9] is true without any restrictions on the parameters α and β.

In [G. Szegö, 1, 8.71, (5)] it is mentioned that the asymptotic formula [G. Szegö,1,(8.71.19)] for Jacobi associated functions can be obtained by the method of the steepest descent. Let us note that it has been already established by G. Darboux [1].

The proof of the asymptotic formula (1.30) is due to I. Baičev [1]. It follows an idea used by N. Obrechkoff in studying the asymptotics of the Bessel polynomials and their associated functions [N. Obrechkoff, 1].

The asymptotic formula (2.1) is proved in [G. Szegö, 1, 8.65] by means of Liouville-Stekloff's method when $z = x$ is real. For the complex case it is only mentioned that:"The proof of Theorem 8.22.7 can be given along these same lines".

The asymptotic expansion (2.1) for the Hermite polynomials and Uspensky's formula [Chapter II, (2.7)] can be used to obtain asymptotic formulas for the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ provided $\alpha \in \mathbb{R}$ [G. Szegö, 1, 8.66]. As it is proved, a version of the asymptotic formula (2.5) ($\nu = 1$) holds even when α is an arbitrary complex number.

The asymptotic formula (2.20) is due to L. Fejér [G. Szegö, 1, (8.22.1)]. Its generalization is given by O. Perron [G. Szegö, 1, (8.22.2)].

A proof that the sequence $\{\mu_n^{(\alpha)}(z)\}_{n=1}^{\infty}$ in the asymptotic formula (3.1) tends uniformly to zero on every compact subset of the region $\mathbb{C} \setminus [0, \infty)$, which is based only on the integral representation [Chapter I,(4.3)], is given in [P. Rusev, 23]. A direct proof of the asymptotic formulas (3.3) and (3.4) which uses only the integral representation [Chapter II, (4.17)] can be found in [P. Rusev, 13].

Inequalities for Jacobi polynomials are obtained by B.C. Carlson [1]. He gave explicitly a sequence of positive numbers $\{f_n(\alpha, \beta)\}_{n=0}^{\infty}, \alpha, \beta \in \mathbb{C}$ such that for $\theta \in \mathbb{C}$ and $n = 0, 1, 2, \ldots,$

$$|P_n^{(\alpha,\beta)}(\cos\theta)| \leq f_n(\alpha, \beta) \exp(n|\operatorname{Im}\theta|),$$

and, moreover, $\lim_{n\to\infty}\{f_n(\alpha,\beta)\}^{1/n} = 1$. This result is quite general since there are no restrictions either on the parameters α, β, or on the complex variable θ. Similar inequalities for the Jacobi associated functions are established by the same author in [B.C. Carlson, 2].

The inequality (4.1) can be found in [G. Szegö, 1, (7.6.11)] and (4.2) is a corollary of Theorem 8.91.1 from the same book. The inequality (4.3) as well as its "generalization" (4.5) in the case when α is an arbitrary complex number is due to P. Rusev [2].

The inequality (4.10) is proved by O. Sz ász [1]. Its O-version has been given by J. Indritz [1]. By the technique used in the proof of (4.3) we come to the inequality (5.1) [P. Rusev, 3].

In a recent paper [P.Rusev, 31] it is proved that the inequality

$$|H_n(z)| \le K_n \exp(x^2)\cosh((2n+1)^{1/2}y),\ z = x+iy,\ n = 0,1,2,\dots,$$

holds with

$$K_n = (2e/\pi)^{1/4}(\Gamma(2n+1))^{1/2}(2n+1)^{-n/2-1/4}\exp(n/2).$$

Chapter IV

CONVERGENCE OF SERIES IN JACOBI, LAGUERRE AND HERMITE SYSTEMS

1. Series in Jacobi polynomials and associated functions

1.1 A series

$$\sum_{n=0}^{\infty} u_n(z) \tag{1.1}$$

of complex-valued functions is called absolutely uniformly convergent on a set $E \subset \overline{\mathbb{C}}$ if the series $\sum_{n=0}^{\infty} |u_n(z)|$ is uniformly convergent on E. Let us note that a series may be absolutely as well as uniformly convergent but not absolutely uniformly convergent.

If the series (1.1) is majorized on E by a convergent series, i.e. if $|u_n(z)| \le \lambda_n,\ z \in E,\ n = 0, 1, 2, \dots$, and $\sum_{n=0}^{\infty} \lambda_n < \infty$ (in such a case the series (1.1) is called normally convergent on E), then it is absolutely uniformly convergent on E. Of course, an absolutely uniformly convergent series, in general, does not need to be normally convergent.

Let G be a subregion of $\overline{\mathbb{C}}$ with a nonempty boundary and let the complex functions $u_n(z),\ n = 0, 1, 2, \dots$ be continuous on the closure of this region and holomorphic in G. Then the absolutely uniformly convergence of the series (1.1) on the boundary of G implies the same property in G.

1.2 Recall that we denoted by $e(r),\ 1 < r < \infty$, the image of the circle $C(0; r)$ by the Zhukovskii transformation $z = (\omega + \omega^{-1})/2$, i.e. $e(r) = \{z \in \mathbb{C} : |\omega(z)| = r\}$, where $\omega(z)$ is that inverse of Zhukovskii function in the region $\overline{\mathbb{C}} \setminus [-1, 1]$, for which $|\omega(z)| > 1$.

The interior of $e(r), 1 < r < \infty$, was denoted by $E(r)$. Further, we define $E(\infty) = \mathbb{C}, E(1) = \emptyset$, $E^*(r) = \overline{\mathbb{C}} \setminus \overline{E(r)},\ 1 < r < \infty, E^*(\infty) = \emptyset$, and $E^*(1) = \mathbb{C} \setminus [-1, 1]$.

(IV.1.1) *Suppose that* $\alpha+1, \beta+1$ *and* $\alpha+\beta+2$ *are not equal to* $0, -1, -2, \dots$.

(a) *If the series*

$$\sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(z) \tag{1.2}$$

converges at a point $\zeta \in \mathbb{C} \setminus [-1,1]$, *then it is absolutely uniformly convergent on every compact subset of the domain* $E(r)$ *with* $r = |\omega(\zeta)|$.

(b) *If* $R = \max\{1, (\limsup_{n\to\infty} |a_n|^{1/n})^{-1}\}$, *then the series* (1.2) *is absolutely uniformly convergent on every compact subset of the domain* $E(R)$ *and diverges in* $E^*(R) \setminus \{\infty\}$.

Proof. **(a)** For each compact set $K \subset E(r)$ there exists $\rho \in (1, r)$ such that $K \subset \overline{E(\rho)}$. Therefore, it is sufficient to prove that the series (1.2) is absolutely uniformly convergent on every ellipse $e(\rho)$ with $1 < \rho < r$. This can be established by means of the asymptotic formula [Chapter III, (1.9)] for the Jacobi polynomials.

Since the series (1.2) converges at the point ζ, there exists a positive constant A such that $|a_n P_n^{(\alpha,\beta)}(\zeta)| \le A$ for each $n = 0, 1, 2, \dots$. Moreover, having in mind **(III.1.3)**, we can assert that the sequence $\{p_n^{(\alpha,\beta)}(z)\}_{n=1}^{\infty}$ tends uniformly to zero on the compact set $e(\rho) \cup \{\zeta\} \subset \mathbb{C} \setminus [-1,1]$. Hence, if $0 < \varepsilon < 1$, then there exists a positive integer ν such that $|p_n^{(\alpha,\beta)}(\zeta)| < \varepsilon$ and $|p_n^{(\alpha,\beta)}(z)| < \varepsilon$ for $z \in e(\rho)$ and $n > \nu$. Define $m(\rho) = \max\{|p^{(\alpha,\beta)}(z)| : z \in e(\rho)\}$. Then [Chapter III, (1.9)] yields

$$|a_n P_n^{(\alpha,\beta)}(z)| = |a_n P_n^{(\alpha,\beta)}(\zeta)| \left| \frac{P_n^{(\alpha,\beta)}(z)}{P_n^{(\alpha,\beta)}(\zeta)} \right| \le A m(\rho) |p^{(\alpha,\beta)}(\zeta)|^{-1} \frac{1+\varepsilon}{1-\varepsilon} \left(\frac{\rho}{r}\right)^n$$

for $z \in e(\rho)$ and $n > \nu$.

Since $1 < \rho < r$, it follows that the series (1.2) is normally and, hence, absolutely uniformly convergent on the ellipse $e(\rho)$.

(b) Suppose that $1 < R < \infty$, i.e. $0 < 1/R = \lambda = \limsup_{n\to\infty} |a_n|^{1/n} < 1$. Let K be a compact subset of $E(R)$ and $r \in (1, R)$ be chosen such that $K \subset E(r)$. Then from [Chapter III, (1.9)] it follows that $\limsup_{n\to\infty} |a_n P_n^{(\alpha,\beta)}(r)|^{1/n} = \lambda r < 1$, i.e. the series (1.2) is (absolutely) convergent at the point r and, hence, by **(a)** it is absolutely uniformly convergent on K.

From [Chapter III, (1.9)] we obtain that

$$\limsup_{n\to\infty} |a_n P_n^{(\alpha,\beta)}(z)|^{1/n} = \lambda |\omega(z)| = R^{-1} |\omega(z)| > 1$$

at each point of the set $E^*(R) \setminus \{\infty\} = \{z \in \mathbb{C} : |\omega(z)| > R\}$, i.e. the series (1.2) diverges in this set.

In the case $R = \infty$, i.e. $\lambda = 0$, we have $\limsup_{n\to\infty} |a_n P_n^{(\alpha,\beta)}(z)|^{1/n} = 0$ at each point of the region $z \in \mathbb{C} \setminus [-1,1]$ and, hence, the series (1.2) converges in this region. Again (a) yields that the series (1.2) is absolutely uniformly convergent on every compact subset of the complex plane.

Let $R = 1$, i.e. $\lambda = 1$. If $z \in \mathbb{C} \setminus [-1,1]$, then

$$\limsup_{n\to\infty} |a_n P_n^{(\alpha,\beta)}(z)|^{1/n} = |\omega(z)| > 1.$$

Hence, in this case the series (1.2) is divergent in the region $E^*(1)$.

1.3 A property like **(IV.1.1)** holds also for series in Jacobi associated functions:

(IV.1.2) *Suppose that* $\alpha+1, \beta+1$ *and* $\alpha+\beta+2$ *are not equal to* $0, -1, -2, \ldots$.
(a) *If the series*

$$\sum_{n=0}^{\infty} b_n Q_n^{(\alpha,\beta)}(z) \tag{1.3}$$

converges at a point $\zeta \in \mathbb{C} \setminus [-1, 1]$, *then it is absolutely uniformly convergent on every closed subset of the region* $E^*(r)$ *with* $r = |\omega(\zeta)|$.

(b) *If* $R = \max\{1, \limsup_{n\to\infty} |b_n|^{1/n}\}$, *then the series* (1.3) *is absolutely uniformly convergent on every closed subset of* $E^*(R)$ *and diverges in* $E(R)\setminus[-1, 1]$.

The proof is similar to that of the previous proposition, and it is based on the asymptotic formula [Chapter III, (1.30)]. We leave it to the reader as an exercise.

2. Series in Laguerre polynomials and associated functions

2.1 Recall that we denoted by $\Delta(\lambda)$, $0 < \lambda < \infty$, the interior of the parabola $p(\lambda) = \{z \in \mathbb{C} : \mathrm{Re}(-z)^{1/2} = \lambda\}$, and by $\Delta^*(\lambda)$ its exterior. Now we define $\Delta(0)=\emptyset$, $\Delta(\infty) = \mathbb{C}$, $\Delta^*(0) = \mathbb{C} \setminus [0, \infty)$ and $\Delta^*(\infty)=\emptyset$.

(IV.2.1) *Let* α *be an arbitrary complex number.*
(a) *If the series*

$$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z) \tag{2.1}$$

converges at a point $\zeta \in \mathbb{C} \setminus [0, \infty)$, *then it is absolutely uniformly convergent on every compact subset of the region* $\Delta(\sigma)$ *with* $\sigma = \mathrm{Re}(-\zeta)^{1/2}$;

(b) *If* $\lambda_0 = \max\{0, -\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n|\}$, *then the series* (2.1) *is absolutely uniformly convergent on every compact subset of the region* $\Delta(\lambda_0)$ *and diverges in* $\Delta^*(\lambda_0)$.

Proof. **(a)** Suppose that K is a compact subset of $\Delta(\sigma)$ and define

$$\lambda = \max\{\mathrm{Re}(-z)^{1/2} : z \in K\}.$$

It is evident that if K is nonempty, then $0 \le \lambda < \sigma$. We choose the nonnegative integer k such that $k + \mathrm{Re}\,\alpha > -1/2$. From **(III.2.3)** it follows that if $0 < \eta < 1$, then there exists a positive integer ν such that the inequality $|\lambda_n^{(\alpha)}(\zeta)| \le \eta$ holds for $z \in K$ and $n > \nu$.

Denote $M = \sup_{n\in\mathbb{N}_0} |a_n L_n^{(\alpha)}(\zeta)|$ and

$$m(\alpha, \zeta) = (2\sqrt{\pi})^{-1}|(-\zeta)^{-\alpha/2-1/4}\exp(\zeta/2)|.$$

If $r = \max\{\operatorname{Re} z : z \in K\}$, then the asymptotic formula [Chapter III, (2.5)] and the inequality [Chapter III, (4.5)] yield

$$|a_n L_n^{(\alpha)}(z)| = |a_n L_n^{(\alpha)}(\zeta)| \left| \frac{L_n^{(\alpha)}(z)}{L_n^{(\alpha)}(\zeta)} \right|$$

$$\leq \frac{MC(\alpha,\lambda,k)\exp r}{m(\alpha,\zeta)(1-\eta)} n^{k+\operatorname{Re}(\alpha/2)+1/4} \exp\{-2(\sigma-\lambda)\sqrt{n}\}$$

provided $z \in K$ and $n > \nu$.

From the above inequality it follows that the series (2.1) is absolutely uniformly convergent on the compact set K.

(b) Suppose that $0 < \lambda_0 < \infty$ and that K is a compact subset of $\Delta(\lambda_0)$. We choose $\delta \in (0, \lambda_0/2)$ such that $K \subset \Delta(\lambda_0 - 2\delta)$. Then from **(III.2.3)** and from the definition of λ_0 it follows that

$$-\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n L_n^{(\alpha)}\{-(\lambda_0-2\delta)^2\}| = 2\delta.$$

From the last equality it follows that there exists a positive integer ν such that the inequality $|L_n^{(\alpha)}\{-(\lambda_0-2\delta)^2\}| \leq \exp(-\delta\sqrt{n})$ holds for $n > \nu$, i.e. the series (2.1) converges at the point $-(\lambda_0-2\delta)^2$. Therefore, it is absolutely uniformly convergent on every compact subset of $\Delta(\lambda_0-2\delta)$ and, in particular, on K.

If $z \in \Delta^*(\lambda_0)$, i.e $\operatorname{Re}(-z)^{1/2} > \lambda_0$, then the definition of λ_0 and the asymptotic formula [Chapter III, (2.6)] yield that

$$\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n L_n^{(\alpha)}(z)| = \operatorname{Re}(-z)^{1/2} - \lambda_0$$

and, hence, the series (2.1) diverges at the point z.

Suppose that $\lambda_0 = \infty$ and let $\zeta \in \mathbb{C} \setminus [0,\infty)$ be arbitrary. Then $-\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n L_n^{(\alpha)}(\zeta)| = \infty$, i.e. the series (2.1) converges at the point ζ and, hence, on each compact subset of $\mathbb{C}$.

Suppose that $\lambda_0 = 0$, i.e. $\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n| \geq 0$. Then

$$\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n L_n^{(\alpha)}(z)| \geq \operatorname{Re}(-z)^{1/2} > 0$$

for $z \in \mathbb{C}\setminus[0,\infty)$, i.e. the series (2.1) diverges in the region $\mathbb{C}\setminus[0,\infty)$.

(IV.2.2) *Suppose that* $\alpha \in \mathbb{C}\setminus\mathbb{Z}^-$ *and*

$$0 < \lambda_0 = -\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n| \leq \infty. \tag{2.2}$$

If $0 \leq \lambda < \lambda_0$ *and* $\rho > \max(1, 2\lambda^2)$, *then the series*

$$\sum_{n=1}^{\infty} a_n z^{\alpha/2+1/4}\exp(-z)L_n^{(\alpha)}(z) \tag{2.3}$$

is absolutely uniformly convergent on the closed set $\tilde{\Delta}(\lambda,\rho) = \overline{\Delta}(\lambda) \cap \{z \in \mathbb{C} : |z| \geq \rho\}$ *where* $\overline{\Delta}(\lambda) = \{z \in \mathbb{C} : \mathrm{Re}(-z)^{1/2} \leq \lambda\}$.

Proof. If $\lambda_0 < \infty$, then we choose $\tau \in (0, \lambda_0 - \lambda)$. Suppose that τ is an arbitrary positive number when $\lambda_0 = \infty$. Then from (2.2) it follows that $|a_n| = O\{\exp(-(2\lambda+\tau)\sqrt{n})\}$ when n tends to infinity. Further, the inequality [Chapter III, (4.3)] gives that

$$|a_n z^{\alpha/2+1/4} \exp(-z) L_n^{(\alpha)}(z)| = O\{n^{\alpha/2-1/4} \exp(-\tau\sqrt{n})\},\ n \longrightarrow \infty,$$

for $z \in \tilde{\Delta}(\lambda,\rho)$, i.e. the series (2.3) is majorized on $\tilde{\Delta}(\lambda,\rho)$ by the series

$$\sum_{n=1}^{\infty} n^{\alpha/2-1/4} \exp(-\tau\sqrt{n}) < \infty. \tag{2.4}$$

2.2 Now we are going to consider the convergence of series in Laguerre associated functions.

(IV.2.3) *Suppose that* $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$.

(a) *If the series*

$$\sum_{n=0}^{\infty} b_n M_n^{(\alpha)}(z) \tag{2.5}$$

is convergent at a point $\zeta \in \mathbb{C} \setminus [0,\infty)$, *then it is absolutely uniformly convergent on every compact subset of the region* $\Delta^*(\sigma)$, *where* $\sigma = \mathrm{Re}(-\zeta)^{1/2}$.

(b) *If* $\mu_0 = \max\{0, \limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|b_n|\}$, *then the series* (2.5) *is absolutely uniformly convergent on every compact subset of the region* $\Delta^*(\mu_0)$ *and diverges in the region* $\Delta(\mu_0) \setminus [0,\infty))$.

This proposition follows from the asymptotic formula [Chapter III, (31)]. A more detailed study of the mode of convergence of the series (2.5) is based on the integral representation [Chapter I, (4.10)].

(IV.2.4) *If* $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-, 0 \leq \mu_0 < \infty$ *and* $\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|b_n| \leq \mu_0$, *then for* $\tau \in \mathbb{R}^+$ *the series* (2.5) *is absolutely uniformly convergent on the closed half-plane* $\overline{H}(\mu_0;\tau) = \{z \in \mathbb{C} : \mathrm{Re}\, z \leq -(\mu_0+\tau)^2\}$.

Proof. There exists a positive constant $B = B(\tau)$ such that

$$|b_n| \leq B \exp((2\mu_0+\tau)\sqrt{n}),\ n = 0,1,2,\ldots.$$

Let $z = x+iy \in \overline{H}(\mu_0;\tau)$ and $n+\alpha > -1$. Then from [Chapter I, (4.10)], [Chapter I, Exercise. 26] and [Chapter III, (3.1)] it follows that

$$|M_n^{(\alpha)}(z)| \leq \int_0^{\infty} \frac{t^{n+\alpha}\exp(-t)}{((t-x)^2+y^2)^{(n+1)/2}}\, dt \leq \int_0^{\infty} t^{n+\alpha}\exp(-t)(t-x)^{n+1}\, dt$$

$$\leq \int_0^\infty \frac{t^{n+\alpha}\exp(-t)}{(t+(\mu_0+\tau)^2)^{n+1}}\,dt = -M_n^{(\alpha)}(-(\mu_0+\tau)^2)$$
$$\leq Mn^{\alpha/2-1/4}\exp(-2(\mu_0+\tau)\sqrt{n}),$$

where M is a positive constant not depending on n. Thus, we obtain that $|b_n M_n^{(\alpha)}(z)| \leq BMn^{\alpha/2-1/4}\exp(-\tau\sqrt{n})$ for $z \in \overline{H}(\mu_0;\tau)$ and $n+\alpha > -1$, and the assertion follows from (2.4).

(IV.2.5) *If $\alpha \in \mathbb{C}\setminus\mathbb{Z}^-$, then*

$$zM_n^{(\alpha)}(z) = M_n^{(\alpha+1)}(z) - M_{n-1}^{(\alpha+1)}(z). \tag{2.6}$$

for $z \in \mathbb{C}\setminus[0,\infty)$ and $n \geq 1$.

Proof. If α is not an integer, then (2.6) follows from [Chapter I, (4.10)], the property of orthogonality of the Laguerre polynomials [Chapter I, (3.13)], and the relation [Chapter I, Exercise 5, (a)]. Indeed,

$$zM_n^{(\alpha)}(z) = \frac{1}{2i\sin\alpha\pi}\int_{p(\lambda)}(-\zeta)^\alpha\exp(-\zeta)L_n^{(\alpha)}(\zeta)\,d\zeta$$
$$+\frac{1}{2i\sin\alpha\pi}\int_{p(\lambda)}\frac{(-\zeta)^\alpha\exp(-\zeta)L_n^{(\alpha)}(\zeta)}{\zeta-z}\,d\zeta$$
$$= -\frac{1}{2i\sin(\alpha+1)\pi}\int_{p(\lambda)}\frac{(-\zeta)^{\alpha+1}\exp(-\zeta)L_n^{(\alpha+1)}(\zeta)}{\zeta-z}\,d\zeta$$
$$+\frac{1}{2i\sin(\alpha+1)\pi}\int_{p(\lambda)}\frac{(-\zeta)^{\alpha+1}\exp(-\zeta)L_{n-1}^{(\alpha+1)}(\zeta)}{\zeta-z}\,d\zeta = M_n^{(\alpha+1)}(z) - M_{n-1}^{(\alpha+1)}(z).$$

If $\alpha = k$ is a nonnegative integer, then (2.6) is a corollary of the integral representation [Chapter I, (4.10)].

As an application of **(IV.2.5)** we obtain the following proposition:

(IV.2.6) *If $\alpha \in \mathbb{R}\setminus\mathbb{Z}^-, 0 \leq \mu_0 < \infty$, and $\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|b_n| \leq \mu_0$, then for $\tau > 0$ and $k = 0,1,2,\dots$ the series*

$$\sum_{n=k+1}^\infty b_n z^{k+1} M_n^{(\alpha)}(z) \tag{2.7}$$

is absolutely uniformly convergent on the half-plane $\overline{H}(\mu_0;\tau)$.

Proof. From **(IV.2.4)**, and relation (2.6) it follows that the series $\sum_{n=1}^\infty b_n z M_n^{(\alpha)}(z)$ is absolutely uniformly convergent on $\overline{H}(\mu_0;\tau)$. Since $z^{k+1}M_n^{(\alpha)}(z)$

$= z^k M_n^{(\alpha+1)}(z) - z^k M_{n-1}^{(\alpha+1)}(z)$, we can proceed further by induction with respect to k.

By means of the inequality [Chapter III, (5.1)] we can prove another assertion for the uniform convergence of series of the kind (2.5):

(IV.2.7) *Suppose that* $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-, 0 \le \mu_0 < \infty$ *and*

$$\limsup_{n\to\infty} (2\sqrt{n})^{-1} \log |b_n| \le \mu_0.$$

If $\alpha > 1/2$, *then the series*

$$\sum_{n=0}^{\infty} b_n(-z)^{-\alpha/2-1/4} M_n^{(\alpha)}(z) \tag{2.8}$$

is absolutely uniformly convergent on every closed domain $\overline{\Delta^*(\mu)}$ *with* $\mu \in (\mu_0, \infty)$. *If* $\alpha \le 1/2$, *then the same is true even for the series* (2.5).

3. Series in Hermite polynomials and associated functions

3.1 Results similar to those for Laguerre polynomials holds for Hermite series, i.e. for series of the kind

$$\sum_{n=0}^{\infty} a_n H_n(z) \tag{3.1}$$

and

$$\sum_{n=0}^{\infty} b_n G_n^{\pm}(z) \tag{3.2}$$

They can be proved by means of the asymptotic formulas and inequalities for the Hermite systems. Another way is to use the relations between the Laguerre and Hermite systems of polynomials and associated functions.

Recall that if $0 \le \tau \le \infty$, then we introduced the denotations $S(\tau) = \{z \in \mathbb{C} : |\operatorname{Im} z| < \tau\}$ [Chapter III, **4.2**] and $S^*(\tau) = \{z \in \mathbb{C} : |\operatorname{Im} z| > \tau\}$, [Chapter III, **5.2**]. We assume $S(0)=\emptyset$, $S(\infty) = \mathbb{C}, S^*(0) = \mathbb{C} \setminus \mathbb{R}$ and $S^*(\infty)=\emptyset$. Then:

(IV.3.1) (a) *If the series* (3.1) *converges at a point* $\zeta \in \mathbb{C} \setminus \mathbb{R}$, *then it is absolutely uniformly convergent on every compact subset of the region* $S(\tau)$ *with* $\tau = |\operatorname{Im} \zeta|$.

(b) *If* $\tau_0 = \max\{0, -\limsup_{n\to\infty}(2n+1)^{-1/2} \log |(2n/e)^{n/2} a_n|\}$, *then the series* (3.1) *is absolutely uniformly convergent on every compact subset of* $S(\tau_0)$ *and diverges in* $S^*(\tau_0)$.

(IV.3.2) *If* $0 < \tau_0 \le \infty$ *and* $-\limsup_{n\to\infty}(2n+1)^{-1/2}\log|(2n/e)^{n/2}a_n| \ge \tau_0$, *then the series*

$$\sum_{n=0}^{\infty} a_n \exp(-z^2)H_n(z) \tag{3.3}$$

is absolutely uniformly convergent on every closed strip $\overline{S}(\tau) = \{z \in \mathbb{C} : |\operatorname{Im} z| \le \tau\}$ *with* $\tau \in [0, \tau_0)$.

(IV.3.3) (a) *If the series* (3.2) *converges at a point* $\zeta \in \mathbb{C} \setminus \mathbb{R}$, *then it is absolutely uniformly convergent on every closed set* $\overline{S^*(\tau)}$ *with* $\tau > |\operatorname{Im}\zeta|$.

(b) *If* $\tau_0 = \max\{0, \limsup_{n\to\infty}(2n+1)^{-1}\log|(2n/e)^{n/2}b_n|\}$, *then for* $\tau \in (\tau_0, \infty)$ *the series* (3.2) *is absolutely uniformly convergent on the closed set* $\overline{S^*(\tau)}$ *and diverges in* $S(\tau_0) \setminus \mathbb{R}$.

3.2 The Jacobi series, i.e. the series of the kind (1.1) and (1.2) are much more like to the power series centered at points of the extended complex plane. That is why it is not surprising that their regions of convergence coincide with their regions of absolute convergence. As for Laguerre and Hermite series is concerned, it seems that "asymptotically" they are related to series of the kind

$$\sum_{n=0}^{\infty} a_n \exp(\pm\lambda_n\zeta), \tag{3.4}$$

where $\{\lambda_n\}_{n=0}^{\infty}$ is an increasing sequence of real and positive numbers. It is well-known that if $\lim_{n\to\infty}\lambda_n^{-1}\log n = 0$, then the region of absolute convergence of the series (3.4) coincides with its region of convergence. This property makes clearer the fact that the same holds for the Laguerre and Hermite series. Of course, this analogy does not go too far. Indeed, there are properties which are typical for the Laguerre and Hermite series only. For instance, an assertion like **(IV.2.7)** is not true, in general, for series of the kind (3.4).

4. Theorems of Abelian type

4.1 Suppose that $\zeta \in \mathbb{C}^*$, i.e. ζ is a non-zero complex number. If $0 \le \theta < \pi/2$ and $0 < \rho < 2|\zeta|\cos\theta$, then denote by $M(\zeta,\rho,\theta)$ the set defined by the inequalities $0 < |\zeta - z| \le \rho$, and $|\arg(1-\zeta^{-1}z)| \le \theta$. A classical property of the power series, usually called (second) Abel's theorem, is the following proposition:

(IV.4.1) *If the power series*

$$\sum_{n=0}^{\infty} a_n z^n \tag{4.1}$$

converges at a point $\zeta \in \mathbb{C}^*$, *then it is uniformly convergent on each set of the kind* $M(\zeta,\rho,\theta)$.

The crucial point of the proof of the above proposition is the fact that the ratio $|\zeta - z|(|\zeta| - |z|)^{-1}$ is bounded on every set of the kind $M(\zeta, \rho, \theta)$, i.e. there exists a constant $K = K(\zeta, \rho, \theta)$ such that $|\zeta - z| \leq K(|\zeta| - |z|)$ for $z \in M(\zeta, \rho, \theta)$.

Suppose that the power series (4.1) has a finite and non-zero radius of convergence R, and let F be the holomorphic function defined by this series in its domain of convergence. Suppose that the power series (4.1) converges at a point $\zeta \in C(0; R)$. Then from **(IV.41)** it follows that for $\theta \in [0, \pi/2)$ and $\rho \in (0, 2R\cos\theta)$,

(4.2), $$\lim_{z \to \zeta, z \in M(\zeta, \rho, \theta)} F(z) = F(\zeta),$$

i.e. the restriction of the function F to every set of the kind $M(\zeta, \rho, \theta)$ is continuous at the point ζ.

4.2 Now we are going to show that properties similar to **(IV.4.1)** hold also for Laguerre and Hermite series. In order to state them, we need some additional notations. If $z_0 = x_0 + iy_0$ is a point of the region $\mathbb{C} \setminus [0, \infty)$ and if, in addition, $y_0 > 0$, then we define $\delta_0 \in (0, \pi/2)$ by the equality $\tan\delta_0 = 2\lambda_0 y_0^{-1}$, where $\lambda_0 = \mathrm{Re}(-z_0)^{1/2}$. If $0 \leq \theta < \pi/2$ and $0 < d < d_0 =$dist$(z_0, [0, \infty))$, then denote by $D(z_0, d, \theta)$ the set defined by the inequalities $0 < |z - z_0| \leq d$, and $|\arg(z_0 - z)$ $-\pi/2 - \delta_0| \leq \theta$.

If $y_0 < 0$, then denote by $D(z_0, d, \theta)$ the image of $D(\overline{z_0}, d, \theta)$ by the map $z \longmapsto \overline{z}$. If $z_0 = -\lambda_0^2 < 0$, then we define $D(-\lambda_0^2, d, \theta)$ by means of the inequalities $0 < |\lambda_0^2 + z| < d < \lambda_0^2$, and $|\arg(\lambda_0^2 + z)| \leq \theta$.

It is clear that for $z_0 \in \mathbb{C} \setminus [0, \infty), 0 < d < d_0$ and $0 \leq \theta < \pi/2$, $D(z_0, d, \theta)$ is a subset of the region $\Delta(\lambda_0) \setminus [0, \infty)$.

If $\mathrm{Im}\, w_0 < 0, 0 \leq \varphi < \pi/2$ and $0 < \eta < \mathrm{Im}\, w_0$, then let $S(w_0, \eta, \varphi)$ be the set defined by the inequalities $\eta \leq \mathrm{Im}\, w < \mathrm{Im}\, w_0$ and $|\arg(w_0 - w) - \pi/2| \leq \varphi$. Obviously, $S(w_0, \eta, \varphi)$ is a subset of the strip $I(w_0) = \{w \in \mathbb{C} : 0 < \mathrm{Im}\, w < \mathrm{Im}\, w_0\}$.

The image of the region $\mathbb{C} \setminus [0, \infty)$ by the map $h : w = i(-z)^{1/2}$ is the upper half-plane. More precisely, the strip $I(w_0)$ with $w_0 = i(-z_0)^{1/2}$ corresponds to the region $\Delta(\lambda_0) \setminus [0, \infty)$. Since this map is univalent, it is conformal. Having in mind the last property as well as that if w_0 with $\mathrm{Im}\, w_0 > 0$ is fixed, then the union of the sets $S(w_0, \eta, \varphi)$, with $0 < \eta < \mathrm{Im}\, w_0$ and $0 \leq \varphi < \pi/2$, is the whole strip $I(w_0)$, we conclude that if $z_0 \in \mathbb{C} \setminus [0, \infty), 0 < d < d_0$, and $0 \leq \theta < \pi/2$, then there exist $0 < \eta < \mathrm{Im}\, w_0$ and $0 \leq \varphi > \pi/2$ such that the image of the set $D(z_0, d, \theta)$ by the map h is contained in $S(w_0, \eta, \varphi)$.

(IV.4.2) **(a)** *If* $\mathrm{Im}\, w_0 > 0, \eta \in (0, \mathrm{Im}\, w_0)$ *and* $\varphi \in [0, \pi/2)$, *then for every* w $\in S(w_0, \eta, \varphi)$,

(4.3) $$|w - w_0| \leq (\cos\varphi)^{-1} |\mathrm{Im}(w - w_0)|.$$

(b) *If $z_0 \in \mathbb{C} \setminus [0,\infty), d \in (0, d_0)$ and $\theta \in [0, \pi/2)$, then there exists a constant K such that for $z \in D(z_0, d, \theta)$,*

$$|z_0 - z| \le K\,\mathrm{Re}\{(-z_0)^{1/2} - (-z)^{1/2}\}. \tag{4.4}$$

Proof. The inequality (4.3) follows directly from the definition of the set $S(w_0, \eta, \varphi)$ or, more precisely, from the requirement $|\arg(w_0 - w) - \pi/2| \le \varphi$.

Suppose that η and φ are chosen such that the set $S(w_0, \eta, \varphi)$ to contain the image of $D(z_0, d, \theta)$ by the map $h : w = i(-z)^{1/2}$. Then (4.3) yields that

$$|z_0 - z| = |w_0^2 - w^2| = |(w_0 - w)(w_0 + w)|$$

$$\le (\cos\varphi)^{-1}|w_0 + w|\,\mathrm{Re}\{(-z_0)^{1/2} - (-z)^{1/2}\}$$

for every $z \in D(z_0, d, \theta)$, and if we define $K = (\cos\varphi)^{-1}\max\{|w_0 + w| : w \in S(w_0, \eta, \varphi)\}$, then (4.4) follows.

(IV.4.3) *Suppose that α is an arbitrary complex number. If the series* (2.1) *is convergent at the point $z_0 \in \mathbb{C} \setminus [0,\infty)$, then it is uniformly convergent on every set of the kind $D(z_0, d, \theta)$.*

Proof. Define $s_\nu(z) = \sum_{n=0}^{\nu} a_n L_n^{(\alpha)}(z)$, $\nu = 0, 1, 2, \dots$. Then there exists $\lim_{\nu\to\infty} s_\nu(z_0) = s$.

From the asymptotic formula [Chapter III, (2.6)] it follows that there exists a positive integer ν_0 such that $L_\nu^{(\alpha)}(z_0) \ne 0$ for $\nu \ge \nu_0$. For such ν we have $(p = 1, 2, 3, \dots)$

$$s_{\nu+p}(z) - s_\nu(z) = -(s_\nu(z_0) - s)\frac{L_{\nu+1}^{(\alpha)}(z)}{L_{\nu+1}^{(\alpha)}(z_0)}$$

$$+ \sum_{n=\nu+1}^{\nu+p-1}(s_n(z_0) - s)\left\{\frac{L_n^{(\alpha)}(z)}{L_n^{(\alpha)}(z_0)} - \frac{L_{\nu+1}^{(\alpha)}(z)}{L_{n+1}^{(\alpha)}(z_0)}\right\} + (s_{\nu+p}(z_0) - s)\frac{L_{\nu+p}^{(\alpha)}(z)}{L_{\nu+p}^{(\alpha)}(z_0)}. \tag{4.5}$$

By means of the asymptotic formula [Chapter III, (2.6)] we easily prove that the series $\sum_{n=\nu_0}^{\infty}\{L_n^{(\alpha)}(z_0)\}^{-1}L_n^{(\alpha)}(z)$ is absolutely convergent in the region $\Delta(\lambda_0)\setminus[0,\infty)$, where $\lambda_0 = \mathrm{Re}(-z_0)^{1/2}$. Therefore, we can define the function $L^{(\alpha)}(z_0; z)$ in this region by the equality

$$L^{(\alpha)}(z_0; z) = \sum_{n=\nu_0}^{\infty}\left|\frac{L_n^{(\alpha)}(z)}{L_n^{(\alpha)}(z_0)} - \frac{L_{n+1}^{(\alpha)}(z)}{L_{n+1}^{(\alpha)}(z_0)}\right|.$$

Substituting $\alpha-1$ for α, and $n+1$ for n in the relation [Chapter III, (2.10)], then from the asymptotic formula [Chapter III,(2.6)] we obtain that for $n \geq \nu_0$ it holds

$$\frac{L_n^{(\alpha)}(z)}{L_n^{(\alpha)}(z_0)} - \frac{L_{n+1}^{(\alpha)}(z)}{L_{n+1}^{(\alpha)}(z_0)} = \frac{L_n^{(\alpha)}(z)}{L_n^{(\alpha)}(z_0)} \cdot \frac{L_{n+1}^{\alpha-1}(z_0)}{L_{n+1}^{(\alpha)}(z_0)} - \frac{L_{n+1}^{(\alpha-1)}(z)}{L_{n+1}^{(\alpha)}(z_0)}$$
$$= n^{-1/2}\exp\{-2((-z_0)^{1/2}-(-z)^{1/2})\sqrt{n}\}l_n^{(\alpha)}(z_0;z),$$

where $\{l_n^{(\alpha)}(z_0;z)\}_{n=\nu_0}^{\infty}$ is a sequence of complex-valued functions which are holomorphic in the region $\mathbb{C}\setminus[0,\infty)$. Moreover, this sequence is uniformly bounded on each compact subset of this region. Since $l_n^{(\alpha)}(z_0;z_0)=0$ for $n \geq \nu_0$, from Schwarz's lemma we obtain that the sequence $\{(z-z_0)^{-1}l_n^{(\alpha})(z_0;z)\}_{n=\nu_0}^{\infty}$ is uniformly bounded on every closed disk $\overline{U}(z_0;d)$, where $0 < d < d_0$. Then, (4.4) yields

$$L^{(\alpha)}(z_0;z) = O\Big(|z_0-z|\sum_{n=\nu_0}^{\infty} n^{-1/2}\exp\{-2\operatorname{Re}((-z_0)^{1/2}-(-z)^{1/2})\sqrt{n}\}\Big)$$
$$= O\Big(|z_0-z|\int_0^{\infty} t^{-1/2}\exp\{-2\operatorname{Re}((-z_0)^{1/2}-(-z)^{1/2})\sqrt{t}\}\,dt\Big)$$
$$= O(|z_0-z|\{\operatorname{Re}((-z_0)^{1/2}-(-z)^{1/2})^{-1}) = O(1)$$

for $z\in D(z_0,d,\theta)$, i.e. the function $L^{(\alpha)}(z_0;z)$ is bounded on each set $D(z_0,d,\theta)$.

Further, from [III, (2.6)] it follows that the sequence $\{L_n^{(\alpha)}(z)/L_n^{(\alpha)}(z_0)\}_{n=\nu_0}^{\infty}$ is also uniformly bounded on each set $D(z_0,d,\theta)$. Then (4.5) yields that $|s_{\nu+p}(z)$ $-s_\nu(z)| = O(|s_\nu(z_0)-s|)$ provided $z\in D)z_0,d,\theta)$, $\nu\geq\nu_0$, $p=1,2,3,\dots$ and, hence, the sequence $\{s_\nu(z)\}_{n=0}^{\infty}$ is uniformly convergent on the set $D(z_0,d,\theta)$.

Denote by $D^*(z_0,d,\theta)$ the set which is symmetric to $D(z_0,d,\theta)$ with respect to the tangent of the parabola $p(\lambda_0)$, $\lambda_0=\operatorname{Re}(-z_0)^{1/2}$ at the point z_0. Then:

(IV.4.4) *If* $\alpha\in\mathbb{R}\setminus\mathbb{Z}^-$ *and the series* (2.5) *is convergent at the point* z_0 $\in\mathbb{C}\setminus[0,\infty)$, *then it is uniformly convergent on every set* $D^*(z_0,d,\theta)$.

4.3 Propositions like **(IV.4.3)** hold also for series in Hermite polynomials as well as in Hermite associated functions. In order to state them, we need to define a set of the kind $S(w_0,\eta,\varphi)$ as the image of the set $S(\overline{w_0},\eta,\varphi)$ by the mapping $w\longrightarrow\overline{w}$ provided $\operatorname{Im}w_0<0$. If $w_0\in\mathbb{C}\setminus\mathbb{R}$, then we define $S^*(w_0,\eta,\varphi)$ as the set which is symmetric to $S(w_0,\eta,\varphi)$ with respect to the line $w\in\mathbb{C}:\operatorname{Im}w=\operatorname{Im}w_0$. Then the following assertions are true:

(IV.4.5) *If the series* (3.1) *is convergent at a point* $w_0\in\mathbb{C}\setminus\mathbb{R}$, *then it is uniformly convergent on every set* $S(w_0,\eta,\varphi)$.

(IV.4.6) *If the series* (3.2) *is convergent at a point* $w_0\in\mathbb{C}\setminus\mathbb{R}$ *then, it is uniformly convergent on every set* $S^*(w_0,\eta,\varphi)$.

5. Uniqueness of the representations by series in Jacobi, Laguerre and Hermite polynomials and associated functions

5.1 Let $\mathcal{E}$ be an infinite dimensional topological vector space over $\mathbb{C}$ and let $X = \{x_n\}_{n=0}^{\infty}$ be a denumerable systems of elements of $\mathcal{E}$. For an element $x \in \mathcal{E}$ it is said that it is representable by the system X or, more precisely, by a series in this system, if there exists a sequence of complex numbers $\{a_n\}_{n=0}^{\infty}$ such that $\lim_{\nu\to\infty} \sum_{n=0}^{\nu} a_n x_n = x$. In such a case we write

$$x = \sum_{n=0}^{\infty} a_n x_n \tag{5.1}$$

and call the numbers $\{a_n\}_{n=0}^{\infty}$ coefficients of the series in (5.1).

Evidently, the set $\mathcal{X}$ of all elements of $\mathcal{E}$ which are representable by the system X is a $\mathbb{C}$-vector subspace of $\mathcal{E}$. Let us note that, in general, $\mathcal{X}$ is not closed in $\mathcal{E}$.

For the system X it is said that it has the uniqueness property if the zero element of $\mathcal{E}$ has unique representation by this system. It is clear that if X has the uniqueness property, then every element of $\mathcal{X}$ has unique representation by the system X, i.e. it is a basis of the space $\mathcal{X}$ as a $\mathbb{C}$-vector subspace of $\mathcal{E}$. Indeed, as it is easy to see, every system having the uniqueness property is linearly independent.

5.2 It is a common trait of the orthogonal systems to have, in general, the uniqueness property. We are going to confirm this for the classical orthogonal polynomials by considering them as denumerable subsystems of appropriate spaces of holomorphic functions.

(IV.5.1) *Suppose that $\alpha+1, \beta+1$ and $\alpha+\beta+2$ are not equal to $0, -1, -2, \dots$ and that $1 < R \le \infty$. If a complex function f has a representation by a pointwise convergent series in the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ in the region $E(R)$, i.e*

$$f(z) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(z), \ z \in E(R), \tag{5.2}$$

then f is holomorphic in $E(R)$ and, moreover, for every $r \in (0, R)$,

$$a_n = \frac{1}{2\pi i I_n^{(\alpha,\beta)}} \int_{e(r)} \varphi(\alpha, \beta; z) P_n^{(\alpha,\beta)}(z) f(z)\, dz, \ n = 0, 1, 2, \dots. \tag{5.3}$$

In particular, if $f \equiv 0$, then $a_n = 0$ for $n = 0, 1, 2, \dots$, i.e. under the assumption that $\alpha+1, \beta+1, \alpha+\beta+2$ are not equal to $0, -1, -2, \dots$ the system of Jacobi's polynomials with parameters α and β has the uniqueness property in each space of the kind $\mathcal{H}(E(R)), 1 < R \le \infty$.

Proof. Since the series on the right-hand side of (5.2) is convergent in the region $E(R)$, from **(IV.1.1)** it follows that it is (absolutely) uniformly convergent on every compact subset of $E(R)$, i.e. by Weierstrass' theorem the function f is holomorphic in $E(R)$.

For every $z \in E(R)$ and $n = 0, 1, 2, \ldots$ we have

$$\varphi(\alpha, \beta; z) P_n^{(\alpha,\beta)}(z) f(z) = \sum_{k=0}^{\infty} \varphi(\alpha, \beta; z) P_n^{(\alpha,\beta)}(z) P_k^{(\alpha,\beta)}(z). \tag{5.4}$$

It is clear that the series in the right-hand side of (5.4) is uniformly convergent on every compact subset of $E(R)$ and, hence, on every ellipse $e(r)$ such that $1 < r < R$. Integrating the equality (5.4) along $e(r)$ and taking into account the orthogonality of the Jacobi polynomials [Chapter I, (3.4),$\gamma = e(r)$], we obtain the coefficient representations (5.3).

Remark. If $\operatorname{Re} \alpha > -1$ and $\operatorname{Re} \beta > -1$, then

$$a_n = \frac{1}{I_n^{(\alpha,\beta)}} \int_{-1}^{1} (1-t)^{\alpha} (1+t)^{\beta} P_n^{(\alpha,\beta)}(t) f(t)\, dt, \ n = 0, 1, 2, \ldots. \tag{5.5}$$

(IV.5.2) *Suppose that* $0 < \lambda_0 \leq \infty$ *and* $\alpha > -1$. *If a complex function* f *has a representation as a pointwise convergent series in the Laguerre polynomials with parameter* α *in the region* $\Delta(\lambda_0)$, *i.e.*

$$f(z) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z), \ z \in \Delta(\lambda_0), \tag{5.6}$$

then f *is holomorphic in* $\Delta(\lambda_0)$ *and for* $n = 0, 1, 2, \ldots,$

$$a_n = \frac{1}{I_n^{(\alpha)}} \int_0^{\infty} t^{\alpha} \exp(-t) L_n^{(\alpha)}(t) f(t)\, dt. \tag{5.7}$$

Proof. Since the series on the right-hand side of (5.6) converges pointwise in the region $\Delta(\lambda_0)$, then due to **(IV.2.1)** it is uniformly convergent on every compact subset of this region and, hence, the function f is holomorphic there. Again **(IV.2.1)** yields the inequality $-\limsup_{n\to\infty} (2\sqrt{n})^{-1} \log |a_n| \geq \lambda_0$, which implies $|a_n| = O\{\exp(-2\lambda)\sqrt{n}\}$ for $\lambda \in (0, \lambda_0)$. Further, if $1 \leq \omega < \infty$ is fixed, then the inequality [Chapter III, (4.2)] gives

$$|a_k t^{\alpha} \exp(-t) L_k^{(\alpha)}(t)| = O\{k^{\alpha/2+1/6} t^{\alpha/2-1/2} \exp(-t/2 - 2\lambda\sqrt{k})\}$$

uniformly with respect to $t \in [\omega, \infty)$ and $k = 1, 2, 3, \ldots$. Then, having in mind [Chapter III, (4.1)], we can assert that the series $\sum_{k=0}^{\infty} a_k t^{\alpha} \exp(-t) L_k^{(\alpha)}(t) L_n^{(\alpha)}(t)$

may be integrated termwise on the interval $(0,\infty)$. Since $t^{\alpha}\exp(-t)L_n^{(\alpha)}(t)f(t)$ is its sum for $t \in (0,\infty)$, the representations (5.7) follow from the orthogonality of the Laguerre polynomials [Chapter I, (3.7)].

(IV.5.3) *Let $0 < \lambda_0 \leq \infty$ and $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$. If the series* (2.1) *is pointwise convergent in $\Delta(\lambda_0)$, and if its sum is identically zero there, then $a_n = 0$ for $n = 0, 1, 2, \ldots$, i.e. the system of Laguerre polynomials with parameter $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$ has the uniqueness property in the space $\mathcal{H}(\Delta(\lambda_0))$.*

Proof. Due to **(IV.2.1)** the series (2.1) is uniformly convergent on every compact subset of the domain $\Delta(\lambda_0)$. By the Weierstrass theorem it can be termwise differentiated and, moreover, all the derivatives of its sum are identically zero in this domain. Then from the relation [Chapter I, Exercise 7], i.e.

$$\{L_n^{(\alpha)}(z)\}' = -L_{n-1}^{(\alpha+1)}(z),\ n = 1, 2, 3, \ldots, \tag{5.8}$$

we can assert that if the assertion holds for the series (2.1), then it remains true also for the series $\sum_{n=0}^{\infty} a_n L_n^{(\alpha-1)}(z)$. That means we need to prove the proposition under consideration only for $\alpha > -1$. But in this case it follows at once from **(IV.5.2)**.

An immediate consequence of **(IV.5.3)** is the following assertion:

(IV.5.4) *Suppose that $0 < \lambda_0 \leq \infty, \alpha \in \mathbb{R}\setminus\mathbb{Z}^-$ and that f is a complex function defined in the region $\Delta(\lambda_0)$. Suppose that for each $\lambda \in (0, \lambda_0)$ the function f can be represented in the region $\Delta(\lambda)$ by a pointwise convergent series in the Laguerre polynomials with parameter α, i.e*

$$f(z) = \sum_{n=0}^{\infty} a_n(\lambda) L_n^{(\alpha)}(z),\ z \in \Delta(\lambda). \tag{5.9}$$

Then f is holomorphic in $\Delta(\lambda_0)$, the coefficients $a_n(\lambda)$, $n = 0, 1, 2, \ldots$ do not depend on λ, i.e $a_n(\lambda) = a_n$, $n = 0, 1, 2, \ldots$, and the expansion (5.9) *holds in $\Delta(\lambda_0)$.*

A proposition like **(IV.5.2)** holds also for the series representations by Hermite polynomials:

(IV.5.5) *If a complex-valued function f is representable in the strip $S(\tau_0)$, $0 < \tau_0 \leq \infty$, by a pointwise convergent series in Hermite polynomials, i.e.*

$$f(z) = \sum_{n=0}^{\infty} a_n H_n(z),\ z \in S(\tau_0), \tag{5.10}$$

then f is holomorphic in $S(\tau_0)$ and, moreover,

$$a_n = \frac{1}{I_n}\int_{-\infty}^{\infty} \exp(-t^2)H_n(t)f(t)\,dt,\ n = 0, 1, 2, \ldots. \tag{5.11}$$

It follows from **(IV.3.1)**, and the inequality [Chapter III,(4.5)]. We leave its proof as an exercise to the reader. As its application we obtain:

(IV.5.6) *The system of Hermite polynomials has the uniqueness property in an arbitrary space of the kind $\mathcal{H}(S(\tau_0)), 0 < \tau_0 \le \infty$.*

5.4 As we have just seen, the orthogonality play the essential role in the proofs of the uniqueness of the expansions in the systems of Jacobi, Laguerre and Hermite polynomials. Now we are going to show that the uniqueness of the representations by the systems of Jacobi, Laguerre and Hermite associated functions is a consequence of their "asymptotic" properties only. We start with the following proposition:

(IV.5.7) *If $\alpha + 1, \beta + 1$ and $\alpha + \beta + 2$ differ from $0, -1, -2, \ldots,$ then for $z \in \mathbb{C} \setminus [-1, 1]$,*

$$zQ_n^{(\alpha,\beta)}(z) = -\frac{1}{2}Q_{n-1}^{(\alpha+1,\beta+1)}(z) - Q_n^{(\alpha,\beta)}(z) + Q_n^{\alpha,\beta+1}(z),\ n = 1, 2, 3, \ldots. \tag{5.12}$$

Proof. We define the open set Ω in the space of the complex variables α and β by the requirement $\alpha+1, \beta+1$ and $\alpha+\beta+2$ to be distinct from $0, -1, -2, \ldots.$ Obviously, the set Ω is open and since it is connected, it is a region in $\mathbb{C}^2$.

If $z \in \mathbb{C} \setminus [-1, 1]$ and $n = 1, 2, 3, \ldots$ are fixed, then $\varphi(\alpha, \beta; z)$ and $P_n^{(\alpha,\beta)}(z)$ are holomorphic in Ω as function of α and β. Then from [I, (4.1)] it follows that the same holds for $Q_n^{(\alpha,\beta)}(z)$ as a function of α and β.

If $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\beta > -1$, then (5.12) follows from the integral representation [Chapter I, (4.4)]. For other "values" of $(\alpha, \beta) \in \Omega$ the assertion follows from the identity theorem for holomorphic functions of several complex variables.

(IV.5.8) *Suppose that $\alpha+1, \beta+1$ and $\alpha+\beta+2$ are not equal to $0, -1, -2, \ldots.$ If $n = 0, 1, 2, \ldots$ is fixed, then*

$$Q_n^{(\alpha,\beta)}(z) = \frac{2^{n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)} z^{-n-1}\{1 + O(z^{-1})\} \tag{5.13}$$

when $z \to \infty$ in the region $\mathbb{C} \setminus [-1, 1]$.

Proof. If $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\beta > -1$, then (5.13) follows from [Chapter I, (4.4)]. For other $(\alpha, \beta) \in \Omega$ the validity of the asymptotic formula (5.13) can be proved by means of relation (5.12).

(IV.5.9) *Suppose that* $\alpha+1, \beta+1$ *and* $\alpha+\beta+2$ *are not equal to* $0, -1, -2, \dots$ *and that* $1 \leq R < \infty$. *If the complex function* f *has the representation*

$$f(z) = \sum_{n=0}^{\infty} b_n Q_n^{(\alpha,\beta)}(z),\ z \in E^*(R), \tag{5.14}$$

then it is holomorphic in $E^*(R)$. *Moreover,*

$$\frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} b_0 = \lim_{z\to\infty} z f(z) \tag{5.15}$$

and

$$\frac{2^{k+\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{\Gamma(k+\alpha+\beta+2)} b_k \tag{5.16}$$

$$= \lim_{z\to\infty} z^{k+1}\Big\{ f(z) - \sum_{n=0}^{k-1} b_n Q_n^{(\alpha,\beta)}(z)\Big\}$$

for $k = 1, 2, 3, \dots$. *In particular, if* $f \equiv 0$, *then* $b_k = 0$ *for* $k = 0, 1, 2, \dots$, *i.e. under the condition that* $\alpha+1, \beta+1$ *and* $\alpha+\beta+2$ *are not equal to* $0, -1, -2, \dots$ *the system of Jacobi associated functions* $\{Q_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ *has the uniqueness property in every space of the kind* $\mathcal{H}(E^*(R)), 1 \leq R < \infty$.

5.5 The above assertion, or more precisely, the representations (5.15) and (5.16) show that the series in Jacobi associated functions have a property similar to that of the asymptotic expansions in the "neighbourhood" of the point at infinity. Now we are to see that a similar property holds for series in Laguerre and Hermite associated functions too.

(IV.5.10) *Suppose that* $0 \leq \mu_0 < \infty$ *and* $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$. *If a complex-valued function* f *has the representation*

$$f(z) = \sum_{n=0}^{\infty} b_n M_n^{(\alpha)}(z),\ z \in \Delta^*(\mu_0), \tag{5.17}$$

then it is holomorphic in $\Delta^*(\mu_0)$. *Moreover, for* $\tau \in \mathbb{R}^+$,

$$\Gamma(\alpha+1) b_0 = \lim_{z\in\overline{H}(\mu_0;\tau), z\to\infty} z f(z) \tag{5.18}$$

and

$$(-1)^k \Gamma(k+\alpha+1) b_k \tag{5.19}$$

$$= \lim_{z\in\overline{H}(\mu_0;\tau), z\to\infty} z^{k+1}\Big\{ f(z) - \sum_{n=0}^{k-1} b_n M_n^{(\alpha)}(z)\Big\},\ k = 1, 2, 3, \dots.$$

In particular, if $f \equiv 0$, *then* $b_k = 0$ *for* $k = 0, 1, 2, \dots,$ *i.e. if* α *is real and not equal to* $-1, -2, -3, \dots,$ *then the system of Laguerre associated functions* $\{M_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ *has the uniqueness property in every space of the kind* $\mathcal{H}(\Delta^*(\mu_0)), 0 \leq \mu_0 < \infty$.

Proof. The fact that f is holomorphic in $\Delta^*(\mu_0)$ is a consequence of **(IV.2.3)**. The representations (5.18) and (5.19) follow immediately from proposition **(IV.2.7)** and the asymptotic formula [Chapter III, (3.2)].

(IV.5.11) *If a complex-valued function* f *has the representation*

$$f(z) = \sum_{n=0}^{\infty} b_n G_n^{\pm}(z), \ z \in S^*(\tau_0), 0 \leq \tau_0 < \infty, \tag{5.20}$$

then it is holomorphic in $S^*(\tau_0)$. *Moreover, for every* $\tau \in (\tau_0, \infty)$,

$$\sqrt{\pi} b_0 = \lim_{z \in \overline{S^*(\tau)}, z \to \infty} z f(z) \tag{5.21}$$

and

$$\sqrt{\pi} b_k = \lim_{z \in \overline{S^*(\tau)}, z \to \infty} z^{k+1} \Big\{ f(z) - \sum_{n=0}^{k-1} b_n G_n^{\pm}(z) \Big\}, \ k = 1, 2, 3, \dots. \tag{5.22}$$

In particular, if $f \equiv 0$, *then* $b_k = 0$ *for* $k = 0, 1, 2, \dots,$ *i.e. the systems of Hermite associated functions has the uniqueness property in every space of the kind* $\mathcal{H}(S^*(\tau_0)), 0 \leq \tau_0 < \infty$.

Proof. The holomorphy of f follows from **(IV.3.3)**. In order to obtain the representations (5.21) and (5.22), we have to prove that a series of the kind $\sum_{n=k}^{\infty} b_n z^{k+1} G_n^{\pm}(z)$, $k = 0, 1, 2, \dots,$ is uniformly convergent on every closed set $\overline{S^*(\tau)}$ with $\tau \in (\tau_0, \infty)$. This follows from **(IV.3.3)** and the equality $zG_n^{\pm}(z) = nG_{n-1}^{\pm}(z) + (1/2)G_{n+1}^{\pm}(z)$, $n = 1, 2, 3, \dots.$ The last one is nothing but the recurrence relation [Chapter I, 4.18)].

Exercises

1. Prove the validity of **(IV.1.2)**.

2. State and prove a proposition similar to **(IV.1.1)** for series of the kind

$$\sum_{n=0}^{\infty} \{a_n P_n^{(\alpha,\beta)}(z) + b_n Q_n^{(\alpha,\beta)}(z)\}.$$

3. Using the inequality [Chapter III, (4.5)], verify the convergence of the series (2.3) when α is an arbitrary complex number.

4. Verify **(IV.2.4)** when $\alpha \in \mathbb{C} \setminus \mathbb{Z}^-$.

5. Prove that **(IV.2.4)** holds even in the case when $\alpha \in \mathbb{C} \setminus \mathbb{Z}^-$.

6. Prove **(IV.3.1)**.

7. Prove **(IV.3.2)**.

8. Prove **(IV.3.3)**.

9. Using the notations of [Chapter III, Exercise 4], prove that if

$$0 < \tau_0 = \min_{k=1,2}\{\max[0, \limsup_{n\to\infty}(2n+1)^{-1/2}\log|(2n/e)a_n^{(k)}|]\} < \infty,$$

then the series

$$\sum_{n=0}^{\infty}\{a_n^{(1)}G_n^{(1)}(z;\tau_0) + a_n^{(2)}G_n^{(2)}(z;\tau_0)\}$$

is absolutely uniformly convergent on every closed strip $\overline{S(\tau)}$, where $0 \le \tau < \tau_0$.

10. Prove **(IV.4.4)**.

11. Prove **(IV.4.5)**.

12. Prove **(IV.4.6)**.

13. Using Exercise 5, prove the uniqueness property of the system $\{M_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ when $\alpha \in \mathbb{C} \setminus \mathbb{Z}^-$.

14. Let the bilinear form $\mathcal{B}(X, Y)$ be defined on a topological vector space. If it is separately continuous with respect to its first argument and the system $\Phi = \{\varphi_n\}_{n=0}^{\infty}$ is orthogonal with respect to $\mathcal{B}$, then Φ has the uniqueness property.

15. Suppose that the series in (5.7) is convergent in the region $\Delta^*(\mu_0)$, $0 \le \mu_0 < \infty$. Prove that:

(a) If for some $\tau > 0$ its sum f has infinitely many zeros in the closed half-plane $\overline{H}(\mu_0;\tau) = \{z \in \mathbb{C} : \operatorname{Re} z \le -(\mu_0+\tau)^2\}$, then $f \equiv 0$;

(b) If $\alpha \le 1/2$ and, moreover, for some $\mu > \mu_0$ the function f has infinitely many zeros in the closed region $\overline{\Delta^*(\mu)}$, then $f \equiv 0$.

Comments and references

It is a common property of series in Jacobi, Laguerre and Hermite polynomials and associated functions that their regions of convergence coincide with their regions of absolute convergence. Moreover, these series are (absolutely) uniformly convergent on the compact subset of their regions of convergence. The properties in question, which are well-known for power and Laurent series, respectively, are corollaries of Abel's lemma for series in classical orthogonal polynomials and associated functions. In addition, we have also formulas of Cauchy-Hadamard's type for these series.

Let us mention that a proposition similar to **(IV.3.3)** is proved by G.G. Walter [1] but for series expansions of complex functions defined by Cauchy type integrals.

The analogy with the power series seems to be deeper, since propositions like Abel's theorem hold true also for series in Laguerre and Hermite systems [P. Rusev, 4], [L. Boyadjiev, 1, 8].

The uniqueness of the representations by series in the systems of Jacobi, Laguerre and Hermite polynomials is obtained, as usually, as a corollary of their orthogonality. The uniqueness of the representations by the systems of Jacobi, Laguerre and Hermite associated functions is proved by means of their asymptotic properties only.

Let us note that a proof of **(IV.5.4)** is given in [P. Rusev, 16, p. 120, Lemma 2]. The proposition **(IV.5.10)** as well as its generalization, i.e. Exercise 15 can be found in [P. Rusev, 14].

The entire functions

$$c_n(z) = A_n \cos[(2n+1)^{1/2}z - n\pi/2],\ n = 0, 1, 2, \dots,$$

$$s_n(z) = A_n \sin[(2n+1)^{1/2}z - n\pi/2],\ n = 0, 1, 2, \dots,$$

where

$$A_{2n} = (-1)^n H_{2n}(0) = \frac{\Gamma(2n+1)}{\Gamma(n+1)},\ n = 0, 1, 2, \dots,$$

$$\begin{aligned} A_{2n+1} &= (-1)^n (4n+3)^{-1/2} H'_{2n+1}(0) \\ &= 2(4n+3)^{-1/2} \frac{\Gamma(2n+2)}{\Gamma(n+1)},\ n = 0, 1, 2, \dots \end{aligned}$$

are introduced in [E. Hille, 2, p.880] and the series

$$(*) \quad C(z) = \sum_{n=0}^{\infty} a_n c_n(z), \qquad (**) \quad S(z) = \sum_{n=0}^{\infty} a_n s_n(z)$$

are called the associated Fourier series to the series (3.1). As it is proved in [E. Hille, 2, 885 - 887] the series (3.1) converges at a point of $\mathbb{C} \setminus \mathbb{R}$ if and only if either the series (*) or (**) converges at such a point. An assertion like this can be regarded as a complex analog of a well-known equiconvergence theorem for series in Hermite polynomials [G. Szegö, 1, Theorem 9.1.5].

An equiconvergence theorem for series in the Laguerre polynomials is given in [L. Boyadjiev, 11, Theorem 3]. He proves that if $-\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n| > 0$, and $-1/2 < \alpha < 1/2$, then the series (2.1) converges at a point of the region $\mathbb{C} \setminus [0, \infty)$ if and only if the series

$$\sum_{n=0}^{\infty} I_n^{(\alpha)} a_n \cos[i(4n+1)\sqrt{-z}]$$

converges at such a point.

Chapter V

SERIES REPRESENTATION OF HOLOMORPHIC FUNCTIONS BY JACOBI, LAGUERRE AND HERMITE SYSTEMS

1. Expansions in series of Jacobi polynomials and associated functions

1.1 The problem of expansion of holomorphic functions in series of Jacobi systems can be solved by means of the Christoffel-Darboux formula for these systems and the asymptotic formulas for the Jacobi polynomials and their associated functions. It is sufficient to consider this problem for holomorphic functions defined by Cauchy type integrals. Recall again that $e(r) = \{z \in \mathbb{C} : |\omega(z)| = r\}$, $1 < r < \infty$, where $\omega(z)$ is that inverse of Zhukovskii function $z = (\omega + \omega^{-1})/2$ in the region $\overline{\mathbb{C}} \setminus [-1,1]$ for which $|\omega(z)| > 1$, and that $E(r)$=int$e(r)$ for $1 < r < \infty$, and $E(\infty) = \mathbb{C}$.

(V.1.1) *Suppose that $\alpha + 1, \beta + 1$ and $\alpha + \beta + 2$ are not equal to $0, -1, -2, \dots, 1 < r < \infty$, and let φ be a L-integrable complex-valued function on the ellipse $e(r)$. Then the function*

$$\Phi(z) = \frac{1}{2\pi i}\int_{e(r)} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta,\ z \in \mathbb{C} \setminus e(r), \tag{1.1}$$

is representable for $z \in E(r)$ by a series of the kind [Chapter IV, (1.2)], *i.e.*

$$\sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(z)$$

with coefficients

$$a_n = \frac{1}{2\pi i I_n^{(\alpha,\beta)}}\int_{e(r)} \varphi(\zeta) Q_n^{(\alpha,\beta)}(\zeta)\, d\zeta,\ n = 0,1,2,\dots. \tag{1.2}$$

Proof. We multiply [Chapter I, (4.28)] by $(2\pi i)^{-1}\varphi(\zeta)$, and integrate along $e(r)$. Thus, we obtain that for $z \in \mathbb{C} \setminus e(r)$

$$\Phi(z) = S_\nu^{(\alpha\beta)}(z) + R_\nu^{(\alpha,\beta)}(z), \tag{1.3}$$

where

$$S_\nu^{(\alpha,\beta)}(z) = \sum_{n=0}^{\nu} a_n P_n^{(\alpha,\beta)}(z),\ \nu = 0,1,2,\dots, \tag{1.4}$$

$$R_\nu^{(\alpha,\beta)}(z) = \frac{1}{2\pi i}\int_{e(r)} \frac{\varphi(\zeta)\Delta_\nu^{(\alpha,\beta)}(z,\zeta)}{\zeta - z}\,d\zeta,\ \ \nu = 0,1,2,\dots,$$

and $\{a_n\}_{n=0}^{\infty}$ are given by the equalities (1.2).

If $z \in \mathbb{C}\setminus[-1,1]$, then the asymptotic formulas [Chapter III, (1.9)], [Chapter III, (1.30)] as well as the Stirling formula yield

$$|R_\nu^{(\alpha,\beta)}(z)| = O\left\{\left(\frac{|\omega(z)|}{r}\right)^{\nu}\int_{e(r)}|\varphi(\zeta)|dm(r)\right\},\ \ \nu\to\infty,$$

where $m(r)$ is the Lebesgue measure on $e(r)$.

If $z \in E(r)\setminus[-1,1]$, then $|\omega(z)| < r$ and $\lim_{\nu\to\infty} R_\nu^{(\alpha,\beta)}(z) = 0$, and (1.4) yields that the function (1.1) has a representation for $z \in E(r)\setminus[-1,1]$ as a series in the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ with coefficients (1.2). If $1 < \rho < r$, then this series is absolutely uniformly convergent on the ellipse $e(\rho)$, hence, in $E(\rho)$, i.e. it represents the function Φ in the whole region $E(r)$.

The Cauchy integral formula and the above assertion yield immediately the following proposition:

(V.1.2) *Suppose that* $\alpha+1, \beta+1, \alpha+\beta+2$ *are not equal to* $0,-1,-2,\dots$ *and that* $1 < R \le \infty$. *Then each function* $f \in \mathcal{H}(E(R))$ *is representable in* $E(R)$ *by a series in the Jacobi polynomials* $\{P_n^{(\alpha\beta)}(z)\}_{n=0}^{\infty}$, *i.e.*

$$f(z) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha),(\beta)}(z), \quad z \in E(R)$$

with coefficients

$$a_n = \frac{1}{2\pi i I_n^{(\alpha,\beta)}}\int_{e(r)} f(\zeta)Q_n^{(\alpha,\beta)}(\zeta)\,d\zeta,\ \ 1<r<R,\ \ n=0,1,2,\dots. \tag{1.5}$$

Suppose that α, β are arbitrary complex number and denote by $\mathcal{H}^{(\alpha,\beta)}(R)$, $1 < R \le \infty$, the $\mathbb{C}$-vector space of all complex-valued functions which are holomorphic in the region $E(R)$ and have there representations by series in Jacobi polynomials with parameters α and β.

(V.1.3) *If* $1 < R \le \infty$ *and* $\alpha+\beta+2 \ne 0,-1,-2,\dots$, *then* $\mathcal{H}^{(\alpha,\beta)}(R) = \mathcal{H}(E(R))$, *i.e. under the assumption on* α *and* β, *just made, the system of Jacobi polynomials* $\{P_n^{(\alpha\beta)}(z)\}_{n=0}^{\infty}$ *is a basis in each of the spaces* $\mathcal{H}(E(R))$, $1 < R \le \infty$.

Proof. Suppose that $\mathcal{H}^{(\alpha+1,\beta+1)}(R) = \mathcal{H}(E(R))$ and let the function f be in the space $\mathcal{P}^{(\alpha+1,\beta+1)}(R)$. Since $f' \in \mathcal{H}(E(R))$, a representation of the kind

$$f'(\zeta) = \sum_{n=0}^{\infty} a_n^{(\alpha+1,\beta+1)}(f')P_n^{(\alpha+1,\beta+1)}(\zeta) \tag{1.6}$$

holds in the region $E(R)$.

The relation [Chapter I, Exercise 4] gives that

$$P_n^{(\alpha+1,\beta+1)}(\zeta) = \frac{2}{n+\alpha+\beta+2}\{P_n^{(\alpha,\beta)}(\zeta)\}',\ n = 0,1,2,\dots,$$

and, hence,

$$f'(\zeta) = \sum_{n=0}^{\infty} \frac{2a_n^{(\alpha+1,\beta+1)}(f')}{n+\alpha+\beta+2}\{P_n^{(\alpha,\beta)}(\zeta)\}',\ \zeta \in E(R).$$

The series in the right-hand side of the above equality is uniformly convergent on each compact subset of the region $E(R)$. Therefore, if $z \in E(R)$, then it can be integrated termwise on the segment $[0,z]$. This gives the representation

$$f(z) = f(0) + \sum_{n=0}^{\infty} \frac{2a_n^{(\alpha+1,\beta+1)}(f')}{n+\alpha+\beta+2}\{P_{n+1}^{(\alpha,\beta)}(z) - P_{n+1}^{(\alpha,\beta)}(0)\}.$$

Since the series in (1.6) is convergent in $E(R)$, from **(IV.1.1),(b)** it follows that for $r \in (0,R)$ we have $|a_n^{(\alpha+1,\beta+1)}(f')| = O(r^{-n})$ when n tends to infinity. As a corollary of [Chapter III, Exercise 7] we obtain that if $1 < \rho < r$, then $|P_{n+1}^{(\alpha,\beta)}(0)| = O(\rho^n),\ n \to \infty$, hence, the series

$$\sum_{n=0}^{\infty} \frac{2a_n^{(\alpha+1,\beta+1)}(f')}{n+\alpha+\beta+2} P_{n+1}^{(\alpha,\beta)}(0)$$

is (absolutely) convergent. Further, if we define

$$a_0^{(\alpha,\beta)}(f) = f(0) - \sum_{n=0}^{\infty} \frac{2a_n^{(\alpha+1,\beta+1)}(f')}{n+\alpha+\beta+2} P_{n+1}^{(\alpha,\beta)}(0),$$

and

$$a_n^{(\alpha,\beta)}(f) = \frac{2a_{n-1}^{(\alpha+1,\beta+1)}(f')}{n+\alpha+\beta+1},\ n = 1,2,3,\dots,$$

then the representation

$$f(z) = \sum_{n=0}^{\infty} a_n^{(\alpha,\beta)}(f) P_n^{(\alpha,\beta)}(z)$$

holds in the region $E(R)$ and, hence, $f \in \mathcal{P}^{(\alpha,\beta)}(R)$.

So far we have proved that if $\mathcal{H}^{(\alpha+1,\beta+1)}(R) = \mathcal{H}(E(R))$, then $\mathcal{H}^{(\alpha,\beta)}(R) = \mathcal{H}(E(R))$. By induction we obtain that if $\mathcal{H}^{(\alpha+k,\beta+k)}(R) = \mathcal{H}(E(R))$ for some integer $k \geq 1$, then $\mathcal{H}^{(\alpha,\beta)}(R) = \mathcal{H}(E(R))$. If $\alpha+\beta+2 \neq 0,-1,-2,\dots$, then there exists a positive integer ν such that $\mathrm{Re}(\alpha+\nu) > -1, \mathrm{Re}(\beta+\nu) > -1$ and, hence, $\mathrm{Re}(\alpha+\beta+2\nu+2) > 0$. Then from **(V.1.2)** it follows that $\mathcal{H}^{(\alpha,\beta)}(R) = \mathcal{H}(E(R))$.

1.2 Now we are going to consider series representations by means of Jacobi associated functions.

(V.1.4) *Suppose that* $\alpha+1, \beta+1, \alpha+\beta+2$ *are not equal to* $0, -1, -2, \dots$, $1 < r < \infty$ *and let* ψ *be an L-integrable complex-valued function on the ellipse* $e(r)$. *Then the function*

$$\Psi(z) = -\frac{1}{2\pi i}\int_{e(r)} \frac{\psi(\zeta)}{\zeta - z}\, d\zeta,\ z \in \overline{\mathbb{C}} \setminus e(r),$$

is representable for $z \in E^*(r)$ *by a series of the kind* [IV, (1.3)] *with coefficients*

$$b_n = \frac{1}{2\pi i I_n^{(\alpha,\beta)}} \int_{e(r)} \psi(\zeta) P_n^{(\alpha,\beta)}(\zeta)\, d\zeta,\ n = 0, 1, 2, \dots. \tag{1.7}$$

The proof proceeds as that of **(V.1.1)**, but by "changing the roles" of ζ and z. The above assertion is true even in the case $r = 1$:

(V.1.5) *Suppose that* $\alpha+1, \beta+1, \alpha+\beta+2$ *are not equal to* $0, -1, -2, \dots$ *and let* b *be an L-integrable complex-valued function on the interval* $[-1, 1]$. *Then the function*

$$B(z) = -\frac{1}{2\pi i}\int_{-1}^{1} \frac{b(t)}{t - z}\, dt,\ z \in \overline{\mathbb{C}} \setminus [-1, 1] = E^*(1) \tag{1.8}$$

is representable for $z \in E^*(1)$ *by a series of the kind* [Chapter IV, (1.3)] *with coefficients*

$$b_n = \frac{1}{2\pi i I_n^{(\alpha,\beta)}} \int_{-1}^{1} b(t) P_n^{(\alpha,\beta)}(t)\, dt,\ n = 0, 1, 2, \dots. \tag{1.9}$$

Proof. Using (1.8) and the Christoffel-Darboux formula [Chapter I, (4.25)], we have that for $z \in E^*(1)$ and $\nu = 0, 1, 2, \dots$,

$$B(z) = \sum_{n=0}^{\nu} b_n Q_n^{(\alpha,\beta)}(z) + K_\nu^{(\alpha,\beta)}(z), \tag{1.10}$$

where

$$K_\nu^{(\alpha,\beta)}(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{b(t)\Delta_\nu^{(\alpha,\beta)}(t, z)}{t - z}\, dt,\ \nu = 0, 1, 2, \dots,$$

and $\Delta^{(\alpha,\beta)}(t, z)$ is obtained from [Chapter I, (4.26)] by substituting t for z and z for ζ.

Further, [Chapter III, Exercise 7], the asymptotic formula [Chapter III, (1.30)] as well as the Stirling formula yield that if $1 < \rho < |\omega(z)|$, then

$$|K_\nu^{(\alpha,\beta)}(z)| = O\left\{ \left(\frac{\rho}{|\omega(z)|}\right)^\nu \int_{-1}^{1} |b(t)(z-t)^{-1}|\, dt \right\}$$

and, hence, $\lim_{\nu\to\infty} K_\nu^{(\alpha,\beta)}(z) = 0$.

Suppose that g is a complex-valued function which is holomorphic in the region $E^*(R), 1 \le R < \infty$. Then for each $r \in (R, \infty)$ and $z \in E^*(R)$ we have

$$g(z) = g(\infty) - \frac{1}{2\pi i} \int_{e(r)} \frac{g(\zeta)}{\zeta - z}\, d\zeta \tag{1.11}$$

which is nothing but the Cauchy integral formula for functions holomorphic in a region of the kind $E^*(R), 1 \le R < \infty$.

The following proposition follows immediately from (1.11) and **(V.1.4)**.

(V.1.6) *Suppose that $\alpha + 1, \beta + 1, \alpha + \beta + 2$ are not equal to $0, -1, -2, \dots$ and that $1 \le R < \infty$. Then every function $g \in \mathcal{H}(E^*(R))$ such that $g(\infty) = 0$ is representable for $z \in E^*(R)$ by a series in the Jacobi associated functions $\{Q_n^{(\alpha,\beta)}(z)\}_{n=0}^\infty$ with coefficients*

$$b_n = \frac{1}{2\pi i I_n^{(\alpha,\beta)}} \int_{e(r)} g(\zeta) P_n^{(\alpha,\beta)}(\zeta)\, d\zeta,\ R < r < \infty,\ n = 0, 1, 2, \dots. \tag{1.12}$$

2. Expansions in series of Hermite polynomials

2.1 Suppose that the function $f \in \mathcal{H}(S(\tau_0)), 0 < \tau_0 \le \infty$, is represented in the strip $z \in S(\tau_0)$ by a series in the Hermite polynomials, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n H_n(z)$$

for $z \in S(\tau_0)$. Then from **(IV.3.1)** it follows that

$$-\limsup_{n\to\infty} (2n+1)^{-1/2} \log|(2n/e)^{n/2} a_n| \ge \tau_0.$$

Therefore, if $0 \le \tau < \tau + \delta < \tau_0$, then $|a_n| = O\{(e/2n)^{n/2} \exp[-(\tau+\delta)\sqrt{2n+1}]\}$, and [III, (4.6)] yields that for $z = x + iy \in \overline{S}(\tau)$

$$|f(z)| \le \sum_{n=0}^{\infty} |a_n H_n(z)| = O\left\{ \exp x^2 \sum_{n=0}^{\infty} \exp(-\delta\sqrt{2n+1}) \right\} = O(\exp x^2). \tag{2.1}$$

Denote by $\mathcal{E}(\tau_0), 0 < \tau_0 \le \infty$, the $\mathbb{C}$-vector space of all complex-valued functions which are holomorphic in the region $S(\tau_0)$ and such that they have there expansions in series of Hermite polynomials. Since the system of Hermite polynomials has the uniqueness property, it is a basis of $\mathcal{E}(\tau_0)$.

From the inequality (2.1) it follows that not any function $f \in \mathcal{H}(S(\tau_0))$ can be represented in the region $S(\tau_0)$ by a series in Hermite polynomials, i.e. $\mathcal{E}(\tau_0)$ is a proper ($\mathbb{C}$-vector) subspace of $\mathcal{H}(S(\tau_0))$. As we shall see later, $\mathcal{E}(\tau_0)$, in general, is not closed in $\mathcal{H}(S(\tau_0))$.

Remark. Suppose that G is a region in $\overline{\mathbb{C}}$ and let $\mathcal{H}(G)$ be the $\mathbb{C}$-vector space of the complex-valued functions holomorphic in G. Usually $\mathcal{H}(G)$ is considered as a topological vector space with respect to the compact-open topology. The last one is generated by the neighborhoods of the zero element of $\mathcal{H}(G)$ of the kind $V(K,\varepsilon) = \{f \in \mathcal{H}(G) : |f(z)| < \varepsilon,\ z \in K\}$ provided that K "runs" the set of the nonempty compact subset of the region G. It is well-known that $\mathcal{H}(G)$ is locally convex, and metrizable as a topological vector space, and that it is complete, i.e. it is a Fréchet space. Usually, the vector subspaces of $\mathcal{H}(G)$ are considered as topological vector spaces with the topology induced by that of $\mathcal{H}(G)$. But a ($\mathbb{C}$-vector) subspace E of $\mathcal{H}(G)$, although it "inherits" the structure of $\mathcal{H}(G)$ as a metrizable space, it may be not complete, in general, i.e. it may be not a Fréchet space.

2.2 The entire functions $\{H_n^*(z)\}_{n=0}^{\infty}$, defined by the equalities

$$H_n^*(z) = I_n^{-1/2} \exp(-z^2/2) H_n(z) \tag{2.2}$$

$$= (\sqrt{\pi} n! 2^n)^{-1/2} \exp(-z^2/2) H_n(z),\ n = 0, 1, 2, \ldots,$$

are called Hermite functions. For $n = 0, 1, 2, \ldots$ the function H_n^* is in the space $L^2(\mathbb{R})$, i.e. H_n^* is square-integrable on the real axis. Moreover, the orthogonal property of the Hermite polynomials yields that

$$\int_{-\infty}^{\infty} H_m^*(t) H_n^*(t)\, dt = \delta_{mn},\ m, n = 0, 1, 2, \ldots, \tag{2.3}$$

i.e. $\{H_n^*\}_{n=0}^{\infty}$ is an orthonormal system in the space $L^2(\mathbb{R})$.

An important property of the Hermite functions is given by the following proposition:

(V.2.1) *If for a complex-valued function* $F \in L^2(\mathbb{R})$

$$A_n(F) = \int_{-\infty}^{\infty} F(t) H_n^*(t)\, dt,\ n = 0, 1, 2, \ldots,$$

then

$$\int_{-\infty}^{\infty} |F(t)|^2\, dt = \sum_{n=0}^{\infty} |A_n(F)|^2. \tag{2.4}$$

In particular, if $A_n(F) = 0,\ n = 0, 1, 2, \dots,$ *then* $F \sim 0$, *i.e.* F *is equal to zero almost everywhere in the interval* $(-\infty, \infty)$.

Propositions similar to that for Hermite polynomials holds also for the system of Hermite functions as well as for the series in this system. In particular:

(V.2.2) **(a)** *If*

$$0 < -\limsup_{n\to\infty}(2n+1)^{-1/2}\log|a_n^*| = \tau_0 \le \infty, \tag{2.5}$$

then the series

$$\sum_{n=0}^{\infty} a_n^* H_n^*(z) \tag{2.6}$$

is absolutely uniformly convergent on every compact subset of the strip $S(\tau_0)$, *i.e. it defines a holomorphic function* f^* *there, and diverges on the set* $S^*(\tau_0)$;

(b) $f^*|\mathbb{R} \in L^2(\mathbb{R})$, *i.e.* $\int_{-\infty}^{\infty} |f^*(t)|^2\, dt < \infty$.

Proof. The part **(a)** follows immediately from the definition of Hermite functions, Stirling's formula and **(IV.3.1)**.

From (2.5) it follows that if $0 < \tau < \tau_0$, then there exists a positive constant $D = D(\tau)$ such that $|a_n^*| \le D\exp(-\tau\sqrt{2n+1}),\ n = 0, 1, 2, \dots.$ Then (2.6) and Schwarz's inequality yield that for $t \in (-\infty, \infty)$,

$$|f^*(t)|^2 \le \left(\sum_{n=0}^{\infty} |a_n^* H_n^*(t)|\right)^2 \le \sum_{n=0}^{\infty} |a_n^*| \sum_{n=0}^{\infty} |a_n^*|(H_n^*(t))^2$$

$$\le D^2 \sum_{n=0}^{\infty} \exp(\tau\sqrt{2n+1}) \sum_{n=0}^{\infty} \exp(-\tau\sqrt{2n+1})(H_n^*(t))^2$$

and, hence,

$$\int_{-\infty}^{\infty} |f^*(t)|^2\, dt \le D^2 \left(\sum_{n=0}^{\infty} \exp(-\tau\sqrt{2n+1})\right)^2 < \infty.$$

(V.2.3) *Suppose that* $0 < \tau_0 \le \infty$ *and let* f^* *be a complex-valued function holomorphic in the strip* $S(\tau_0)$. *Suppose that* $f^*|\mathbb{R} \in L^2(\mathbb{R})$ *and denote*

$$a_n^*(f^*) = \int_{-\infty}^{\infty} f^*(t) H_n^*(t)\, dt,\ n = 0, 1, 2, \dots. \tag{2.7}$$

Then in order that f^* *be representable in* $S(\tau_0)$ *by a series in Hermite functions, it is necessary and sufficient that*

$$-\limsup_{n\to\infty}(2n+1)^{-1/2}\log|a_n^*| \ge \tau_0. \tag{2.8}$$

Proof. If the function f^* has an expansion in a series of Hermite functions in $S(\tau_0)$, then its coefficients are given by the equalities (2.7) and the necessity follows from part **(a)** of **(V.2.2)**.

Suppose that (2.8) holds. Then again from **(V.2.2)** it follows that the series $\sum_{n=0}^{\infty} a_n^*(f^*)H_n^*(z)$ defines a function $\tilde{f}$ holomorphic in the strip $S(\tau_0)$ and, moreover, $\tilde{f}|\mathbb{R} \in L^2(\mathbb{R})$.

Further, for each $n = 0, 1, 2, \ldots$ we have the equality

$$\int_{-\infty}^{\infty} \tilde{f}(t)H_n^*(t)\,dt = a_n^*(f^*) = \int_{-\infty}^{\infty} f^*(t)H_n^*(t)\,dt,$$

i.e.

$$\int_{-\infty}^{\infty} \{\tilde{f}(t) - f^*(t)\}H_n^*(t)\,dt = 0,\ n = 0, 1, 2, \ldots.$$

Since the function $\tilde{f} - f^*$ is continuous, from **(V.2.1)** it follows that $\tilde{f}(t) = f^*(t)$ for $t \in \mathbb{R}$ and the identity theorem for holomorphic functions gives that $f^* \equiv \tilde{f}$, i.e. f^* has an expansion in a series of Hermite functions in the strip $S(\tau_0)$.

2.3 The differential equation [Appendix, (4.1)], with z replaced by $z\sqrt{2}$, becomes

$$w'' + (2\nu + 1 - z^2)w = 0. \tag{2.9}$$

It is well-known [E.T. Whittaker, G.N. Watson, 1, 16.5] that the function h_ν^*, defined by the equality

$$h_\nu^*(z) = 2^{\nu/2}D_\nu(z\sqrt{2}),\ z, \nu \in \mathbb{C}, \tag{2.10}$$

is the unique solution of the equation (2.9) such that

$$\lim_{|\arg z| \le 3\pi/4,\, z\to\infty} \exp(z^2/2)(2z)^{-\nu}h_\nu^*(z) = 1. \tag{2.11}$$

From the definition of Hermite functions and relation [Chapter I, (5.16)] it follows that

$$H_n^*(z) = I_n^{-1/2}h_n^*(z),\ n = 0, 1, 2, \ldots. \tag{2.12}$$

The functions $h_{-\nu-1}(\pm iz)$ are also solutions of the equation (2.9) and if $\nu \in \mathbb{C} \setminus \mathbb{Z}^-$, then [Appendix, (4.3)]

$$h_\nu^*(z) = \pi^{-1/2}\Gamma(\nu + 1)2^\nu\{\exp(\nu\pi i/2)h_{-\nu-1}^*(iz) + \exp(-\nu\pi i/2)h_{-\nu-1}^*(-iz)\}.$$

As a corollary of (2.12) we obtain that

$$H_n^*(z) = \pi^{-1} I_n^{1/2}\{i^n h_{-n-1}^*(iz) + (-i)^n h_{-n-1}^*(-iz)\},\ n = 0, 1, 2, \dots . \tag{2.13}$$

We define the functions $\xi_n(z),\ n = 0, 1, 2, \dots,$ on the (closed) upper half-plane $\operatorname{Im} z \geq 0$ by the equalities

$$\xi_n(z) = -(\pi/4)N^2 + \int_0^z (N^2 - \zeta^2)\, d\zeta,\ n = 0, 1, 2. \dots, \tag{2.14}$$

where $N = (2n+1)^{1/2}$, provided $(N^2 - t^2)^{1/2} \geq 0$ when $-N \leq t \leq N$.

Denote by $S_n(\delta),\ n = 0, 1, 2, \dots$ the region defined by the inequalities: $\operatorname{Im} z > 0, -N < \operatorname{Re} z < N$, and $|z \pm N| > \delta$ provided $0 < \delta < 1$. Then [E. Hille, 1, (3.11)]:

(V.2.4) *The representation*

$$\begin{aligned} &2N^{n+1} \exp\{-N^2(1 + \pi i)/4\} h_{-n-1}^*(-iz) \\ &= (1 - z^2/n^2)^{-1/4} \exp(i\xi_n(z))(1 + \eta_n(z)) \end{aligned} \tag{2.15}$$

holds in the region $S_n(\delta)$. Moreover, there exist $M = M(\delta)$ and $n_0 = n_0(\delta)$ such that $|\eta_n(z)| \leq M$ for $z \in S_n(\delta)$ and $n > n_0$.

(V.2.5) *Denote by $E^+(n, \tau),\ 0 < \tau < \infty,\ n = 0, 1, 2, \dots$ the arc of the ellipse $E(n, \tau) :\ x^2/n^2 + y^2/\tau^2 = 1$ in the upper half-plane. Then the inequality*

$$\operatorname{Im} \xi_n(x + iy) + |x|(\tau^2 - y^2)^{1/2} \geq \tau N - (5/24)\tau^3 \tag{2.16}$$

holds for $z = x + iy \in E^+(n, \tau)$.

Proof. If $z = x + iy$ and $y > 0$, then as a path of integration in (2.14) we choose the union of the segments $[0, x]$, and $[x, x + iy]$. Then from (2.14) it follows that

$$\operatorname{Im} \xi_n(x+iy) = \operatorname{Im}\left\{\int_{[x,x+iy]} (N^2 - \zeta^2)^1/2\, d\zeta\right\} = \operatorname{Re}\left\{\int_0^y (N^2 - x^2 - 2ixt + t^2)^{1/2}\, dt\right\}.$$

Substituting $a^2 = N^2 - x^2, b^2 = -2ixt + t^2$ in the identity

$$(a^2 + b^2)^{1/2} - a - \frac{b}{2a} = -\frac{b^2}{4a^2}\left\{(a^2 + b^2)^{1/2} + a + \frac{b}{2a}\right\}^{-1}$$

we obtain that

$$\begin{aligned} &(N^2 - x^2 - 2ixt + t^2)^{1/2} \\ &= (N^2 - x^2)^{1/2} + (-ixt + t^2/2)(N^2 - x^2)^{-1/2} - P(x, t)/Q(n, x, t), \end{aligned}$$

where $P(x,t) = t^2(t-2ix)^2$ and

$$Q(n,x,t) = 4(N^2-x^2)\{(N^2-x^2-2ixt+t^2)^{1/2}$$

$$+(N^2-x^2)^{1/2} + (t^2/2-ixt)(N^2-x^2)^{-1/2}\}.$$

Since

$$|Q(n,x,t)| \geq \operatorname{Re} Q(n,x,t) = 4(N^2-x^2)\{2^{-1/2}[N^2-x^2+t^2$$

$$+((N^2-x^2+t^2)^2+4x^2t^2)^{1/2}]^{1/2} + (N^2-x^2)^{1/2} + (N^2-x^2)^{-1/2}(t^2/2)\}$$

and $\{\operatorname{Re} Q(n,x,t)\}'_t \geq 0$ when $0 \leq t \leq y$, we find that $|Q(n,x,t)| \geq \operatorname{Re} Q(n,x,0)$, i.e. $|Q(n,x,t)| \geq 8(N^2-x^2)^{3/2}$ when $0 \leq t \leq y$.

Further,

$$\operatorname{Im} \xi_n(x+iy) = \operatorname{Re}\left\{\int_0^y [(N^2-x^2)^{1/2} + (-ixt+t^2/2)(N^2-x^2)^{-1/2}\right.$$

$$\left.-P(x,t)/Q(n,x,t)]\,dt\right\} = (N^2-x^2)^{1/2}y + (1/6)(N^2-x^2)^{-1/2}y^3 + \rho(n,x,y),$$

where

$$\rho(n,x,y) = \operatorname{Re}\left\{-\int_0^y [P(x,t)/Q(n,x,t)]\,dt\right\}.$$

Since $|P(x,t)| \leq t^2|t-2ixt|^2 \leq 5N^2t^2$ when $-N \leq x \leq N$,

$$|\rho(n,x,y)| \leq \int_0^y |P(x,t)/Q(n,x,t)|\,dt \leq (5/24)N^2(N^2-x^2)^{-3/2}y^3.$$

If $-N < x < N$ and $0 < y \leq \tau$, then

$$\operatorname{Im} \xi_n(x+iy) + |x|(\tau^2-y^2)^{1/2}$$

$$\geq |x|(\tau^2-y^2) + (N^2-x^2)^{1/2}y - (5/24)N^2(N^2-x^2)^{-3/2}y^3.$$

But if $x+iy \in E^+(n,\tau)$, then $|x| = (N/\tau)(\tau^2-y^2)^{1/2}$, $(N^2-x^2)^{1/2} = (N/\tau)y$ and (2.16) follows.

2.4 Suppose that the function $f \in \mathcal{E}(\tau_0), 0 < \tau_0 \leq \infty$, i.e. that it has a representation of the kind [Chapter IV,(4.9)] in the strip $S(\tau_0)$. If we define $f^*(z) = \exp(-z^2/2)f(z)$, $a_n^* = I_n^{1/2}a_n$, $n = 0,1,2,\ldots$, then f^* has an expansion of the kind

$$f^*(z) = \sum_{n=0}^{\infty} a_n^* H_n^*(z) \tag{2.17}$$

in the strip $S(\tau_0)$.

Conversely, if for a function $f^* \in \mathcal{H}(S(\tau_0))$ the representation (2.17) holds, then the function $f(z) = \exp(z^2/2)f^*(z)$ has an expansion in Hermite polynomials in the strip $S(\tau_0)$ and, hence, $f \in \mathcal{E}(\tau_0)$. It turns out that in order to have a growth characteristic of the functions of the space $\mathcal{E}(\tau_0)$, it is sufficient to know the growth of the holomorphic functions which are representable by series in Hermite functions in the region $S(\tau_0)$.

(V.2.6) *A function* $f^* \in \mathcal{H}(S(\tau_0))$, $0 < \tau_0 \le \infty$, *is representable in the strip* $S(\tau_0)$ *by a series in Hermite functions if and only if for each* $\tau \in [0, \tau_0)$ *there exists a constant* $B^*(\tau)$ *such that the inequality*

$$|f^*(z)| \le B^*(\tau)\exp\{-|x|(\tau^2 - y^2)^{1/2}\} \tag{2.18}$$

holds for $z = x + iy \in \overline{S}(\tau)$.

Proof. Suppose that the representation (2.17) holds in the strip $S(\tau_0)$, $0 < \tau_0 \le \infty$. Then for each $\sigma \in (0, \tau_0)$,

$$\sum_{n=0}^{\infty} |a_n^*|^2 \exp(2\sigma N) = A^2(\sigma) < \infty. \tag{2.19}$$

Indeed, since the series in the right-hand side of (2.17) is supposed to be convergent in the strip $S(\tau_0)$, $-\limsup_{n\to\infty}(2n+1)^{-1/2}\log|a_n^*| \ge \tau_0$. Therefore, if $\sigma < \sigma' < \tau_0$, then there exists a positive integer $n_0 = n_0(\sigma')$ such that $|a_n^*| \le \exp\{-\sigma'(2n+1)^{1/2}\}$ when $n > n_0$. This implies the inequality $|a_n^*|^2 \exp(2\sigma N) \le \exp\{-2(\sigma' - \sigma)\}$ and (2.19) follows from it.

If $0 \le \tau < \sigma < \tau_0$, then the series

$$H(\sigma; x, y) = \sum_{n=0}^{\infty} |H_n^*(x+iy)|^2 \exp(-2\sigma N) \tag{2.20}$$

converges for $x + iy \in \overline{S}(\tau)$. Indeed, the definition of the Hermite functions by (2.2), the inequality [Chapter III, (4.6)] and the Stirling formula yield that if $z = x + iy \in \overline{S}(\tau)$, then $|H_n^*(x+iy)| = O\{\exp(x^2 + \tau\sqrt{2n+1})\}$.

Further, (2.19), (2.20), and Schwarz's inequality as well as the expansion (2.17) yield that if $0 \le \tau < \sigma < \tau_0$, then for $z = x + iy \in \overline{S}(\tau)$,

$$|f^*(z)|^2 = |f^*(x+iy)|^2 \le A^2(\sigma)H(\sigma; x, y). \tag{2.21}$$

Remark. Let $\sigma = (\tau + \tau_0)/2$ if $\tau_0 < \infty$ and let $\sigma = \tau + 1$ if $\tau_0 = \infty$.

Define for $p, q > 0$ and $\nu \in \mathbb{R}$

$$J(p, q, \nu) = \int_0^{\infty} \exp(-pt - q/t)t^{\nu}\, dt. \tag{2.22}$$

Then, in particular,

$$J(p,q,3/2)=\pi^{1/2}q^{-1/2}\exp\{-2(pq)^{1/2}\} \tag{2.23}$$

and, since $J(p,q,5/2)=-J_q'(p,q,3/2)$,

$$J(p,q,5/2)=\pi^{1/2}q^{-3/2}\{1/2+(pq)^{1/2}\}\exp\{-2(pq)^{1/2}\}. \tag{2.24}$$

Further, using (2.23), (2.24), and Schwarz's inequality we obtain that

$$J(p,q,2)\le \pi^{1/2}q^{-1}\{1/2+(pq)^{1/2}\}\exp\{-2(pq)^{1/2}\}. \tag{2.25}$$

From (2.23) it follows that $\exp(-2\sigma N)=\pi^{-1/2}\sigma J(2n+1,\sigma^2,3/2)$ $=\pi^{-1/2}\sigma\int_0^\infty \exp[-(2n+1)t-\sigma^2/t]t^{-3/2}\,dt$. Then for $z=x+iy\in\overline{S}(\tau)$,

$$H(\sigma;x,y)=\sum_{n=0}^\infty |H_n^*(x+iy)|^2\pi^{-1/2}\sigma\int_0^\infty \exp[-(2n+1)t-\sigma^2/t]t^{-3/2}\,dt$$

$$=\pi^{-1/2}\sigma\int_0^\infty \exp(-\sigma^2/t)t^{-3/2}\Big\{\sum_{n=0}^\infty |H_n^*(x+iy)|^2\exp[-(2n+1)t]\Big\}\,dt.$$

Putting $\zeta=\overline{z}=x-iy$ in [Chapter II, (3.6)] and having in mind (2.2), we obtain that if $|w|<1$, then

$$\sum_{n=0}^\infty |H_n^*(x+iy)|^2w^n=\pi^{-1/2}(1-w^2)^{-1/2}\exp\Big\{-\frac{1-w}{1+w}x^2+\frac{1+w}{1-w}y^2\Big\}$$

Setting $w=\exp(-2t),\ 0<t<\infty$, we obtain that

$$H(\sigma;x,y)$$

$$=\pi^{-1}\sigma\int_0^\infty (1-\exp(-4t))^{-1/2}\exp(-\sigma^2/t-t-x^2\tanh t+y^2\coth t)t^{-3/2}\,dt.$$

But $\coth t\le(1+t)/t$ and $-\tanh t\le -t/(1+t)$ for $t<0$; hence,

$$(\pi/\sigma)H(\sigma;x,y)\le\int_0^\infty (1-\exp(-4t))^{-1/2}\exp(-\sigma^2/t-t-x^2t/(1+t)$$

$$+y^2(1+t)/t)t^{-3/2}\,dt=\exp y^2.H^*(\sigma;x,y),$$

where

$$H^*(\sigma;x,y)=\int_0^\infty h^*(\sigma;x,y,t)\,dt$$

and

$$h^*(\sigma; x, y, t) = (1 - \exp(-4t))^{-1/2} \exp\Big\{-\frac{\sigma^2 - y^2}{t} - t - \frac{x^2 t}{1+t}\Big\} t^{-3/2}.$$

Suppose that $|x| \geq 4\sigma$ and denote $\xi = 4\sigma|x|^{-1}$ as well as

$$H_1^*(\sigma; x, y) = \int_0^\xi h^*(\sigma; x, y, t)\, dt,$$

and

$$H_2^*(\sigma; x, y) = \int_\xi^\infty h^*(\sigma; x, y, t)\, dt.$$

Since $(1 - \exp(-u))^{-1} \leq u^{-1} + 1$ for $u > 0$, it follows that if $t > 0$, then

$$(1 - \exp(-4t))^{-1/2} \leq (1/(4t) + 1)^{1/2} \leq (1/2)t^{-1/2} + 1. \tag{2.26}$$

The function $t/(1+t)$ increases in the interval $[0, \infty)$. Hence, it attains its minimum in the interval $[\xi, \infty)$ at the point $t = \xi$, and this minimum is $\xi(1+\xi)^{-1}$ $= 4\sigma(4\sigma + |x|)^{-1}$. Then (2.26) and (2.22) yield that

$$H_2^*(\sigma; x, y) \leq \exp\Big\{-\frac{4\sigma x^2}{4\sigma + |x|}\Big\}\{J(1, \sigma^2 - y^2, 3/2) + (1/2)J(1, \sigma^2 - y^2, 2)\}.$$

Since $|x| \geq 4\sigma$, from (2.23) and (2.24) it follows that

$$\exp[2\sigma|x| + 2(\sigma^2 - y^2)^{1/2}] H_2^*(\sigma; x, y)$$

$$\leq \pi^{1/2}(\sigma^2 - y^2)^{-1}\{(1/2)[1/2 + (\sigma^2 - y^2)^{1/2}]^{1/2} + (\sigma^2 - y^2)^{1/2}\}.$$

Then the choice of σ and the fact that $|y| \leq \tau$ lead to the conclusion that there exists a constant $B_2^*(\tau)$ such that

$$H_2^*(\sigma; x, y) \leq B_2^*(\tau) \exp(-2\tau|x|) \leq B_2^*(\tau) \exp\{-2|x|(\tau^2 - y^2)^{1/2}\}.$$

The above inequality was proved under the assumption $|x| \geq 4\sigma$, but a suitable choice of the constant $B_2^*(\tau)$ ensure its validity for every $x \in \mathbb{R}$.

The function $t^{1/2}(1 - \exp(-4t))^{-1/2}$ is increasing in the interval $(0, \infty)$ and since $(1 - \exp(-4))^{-1/2} < 2$, we have $(1 - \exp(-4t))^{-1/2} < 2t^{-1/2}$ for $t \in (0, 1)$. Moreover, $-t/(1+t) = -t + t^2/(1+t) < -t + t^2$ for $t > 0$ and since $0 < \xi \leq 1$, we obtain

$$H_1^*(\sigma; x, y)$$

$$\leq 2\exp(\xi^2 x^2) \int_0^\xi \exp\{-(\sigma^2 - y^2)/t - x^2 t\} t^{-2}\, dt < 2\exp(16\sigma^2) J(x^2, \sigma^2 - y^2, 2)$$

$$\leq 2\pi^{1/2}\exp(16\sigma^2)(\sigma^2-y^2)^{-1}[1/2+|x|(\sigma^2-y^2)^{1/2}]^{1/2}\exp\{-2|x|(\sigma^2-y^2)^{1/2}\}.$$

The function

$$b(\tau;x,y)$$
$$=(\sigma^2-y^2)^{-1}[1/2+|x|(\sigma^2-y^2)^{1/2}]^{1/2}\exp\{-2|x|[(\sigma^2-y^2)^{1/2}-(\tau^2-y^2)^{1/2}]\}$$

is bounded on the (closed) strip $\overline{S}(\tau)$ and, hence, if

$$B_1^*(\tau)=2\pi^{1/2}\exp(16\sigma^2)\sup\{b(\tau;x,y):x+iy\in\overline{S}(\tau)\},$$

then

$$H_1^*(\sigma;x,y)\leq B_1^*(\tau)\exp\{-2|x|(\tau^2-y^2)^{1/2}\}$$

provided that $x+iy\in\overline{S}(\tau)$.

So far we have proved that if the function $f^*\in H(S(\tau_0))$, $0<\tau_0\leq\infty$, has a representation by a series in Hermite functions in the strip $S(\tau_0)$, then for every $\tau\in[0,\tau_0)$ there exists a constant $B^*(\tau)$ such that the inequality (2.18) holds for $z=x+iy\in\overline{S}(\tau)$.

Conversely, suppose that for a function $f^*\in\mathcal{H}(S(\tau_0))$, $0<\tau_0\leq\infty$, an inequality of the kind (2.18) holds in each closed strip $\overline{S}(\tau)$ with $\tau\in[0,\tau_0)$. In particular, if $0<\tau<\tau_0$, then

$$|f^*(t)|=O\{\exp(-\tau|t|)\},\ -\infty<t<\infty, \tag{2.27}$$

hence, $f^*|\mathbb{R}\in L^2(\mathbb{R})$. Having in mind **(V.2.3)**, we can assert that in order to prove that the function f^* is representable by a series in Hermite functions in the strip $S(\tau_0)$, we have to show that the inequality (2.8) holds for the sequence $\{a_n^*(f^*)\}_{n=0}^{\infty}$ which is defined by the equalities (2.7). In other words we have to prove that for $\tau\in(0,\tau_0)$,

$$|a_n^*(f^*)|=O\{\exp(-\tau N)\}. \tag{2.28}$$

To that end we define $(n=0,1,2,\dots)$

$$a_{n,1}^*(f^*)=\int_{-N+1}^{N-1}f^*(t)H_n^*(t)\,dt$$

and

$$a_{n,2}^*(f^*)=\int_{-\infty}^{-N+1}f^*(t)H_n^*(t)\,dt+\int_{N-1}^{\infty}f^*(t)H_n^*(t)\,dt.$$

From inequality [Chapter III, (4.5)] it follows that the sequence of Hermite functions is uniformly bounded on the interval $(-\infty,\infty)$, and then the inequality (2.27) gives that for $\tau\in(0,\tau_0)$,

$$|a_{n,2}^*(f^*)|=O\left(\int_{N-1}^{\infty}\exp(-\tau t)\,dt\right)=O\{\exp(-\tau N)\}. \tag{2.29}$$

It remains to show that for $\tau \in (0, \tau_0)$,

$$|a^*_{n,1}(f^*)| = O\{\exp(-\tau N)\}. \tag{2.30}$$

From (2.13) we obtain

$$a^*_{n,1}(f^*)$$
$$= \pi^{-1} I_n^{1/2} \left\{ i^n \int_{-N+1}^{N-1} f^*(t) h^*_{-n-1}(it)\, dt + (-i)^n \int_{-N+1}^{N-1} f^*(t) h^*_{-n-1}(-it)\, dt \right\}$$
$$= \pi^{-1} I_n^{1/2} \int_{-N+1}^{N-1} f^*_n(t) h^*_{-n-1}(-it)\, dt,$$

where $f^*_n(z) = i^n f^*(-z) + (-i)^n f^*(z),\ z \in S(\tau_0)$.

Denote by P_n (Q_n) the point of intersection of the line $\operatorname{Re} z = -N+1$ $(\operatorname{Re} z = N-1)$ and the arc $E^+(n,\tau)$ of the ellipse $E(n,\tau)$. Then Cauchy's integral theorem yields that

$$a^*_{n,1}(f^*) = \pi^{-1} I_n^{1/2} \left\{ \int_{[-N+1, P_n]} f^*_n(z) h^*_{-n-1}(-iz)\, dz \right.$$
$$\left. + \int_{\widehat{P_n Q_n}} f^*_n(z) h^*_{-n-1}(-iz)\, dz + \int_{[Q_n, N-1]} f^*_n(z) h^*_{-n-1}(-iz)\, dz \right\}.$$

From the asymptotic formula (2.15) and Stirling's formula it follows that

$$I_n^{1/2} |h^*_{-n-1}(-iz)| = O\{n^{-1/4} |1 - z^2/N^2|^{-1/4} \exp(-\operatorname{Im} \xi_n(z))\} \tag{2.31}$$

in the region $S_n(\delta)$.

Then, having in mind that $\operatorname{Im} \xi_n(z) > 0$ for $z \in S_n(\delta)$ as well as that

$$|1 - z^2/N^2| = O(n^{1/8}),\ |f^*_n(z)| = O\{\exp(-\tau N)\}$$

for $z \in \{[-N+1, P_n] \bigcup [Q_n, N-1]\}$, and that the lengths of the segments $[-N+1, P_n], [Q_n, N-1]$ are $O(n^{-3/4})$ when $n \to \infty$, we obtain

$$I_n^{1/2} \left| \int_{[-N+1, P_n]} f^*_n(z) h^*_{-n-1})(-iz)\, dz \right| = O(n^{-7/8} \exp(-\tau N))$$

and

$$I_n^{1/2} \left| \int_{[Q_n, N-1]} f^*_n(z) h^*_{-n-1}(-iz)\, dz \right| = O(n^{-7/8} \exp(-\tau N)).$$

Since $\displaystyle\int_{\widehat{P_n Q_n}} |1 - z^2/N^2|^{-1/4}\, ds = O(N)$, (2.16), (2.18) and (2.31) yield that

$$I_n^{1/2} \left| \int_{\widehat{P_n Q_n}} f^*_n(z) h^*_{-n-1}(-iz)\, dz \right| = O\{n^{1/4} \exp(-\tau N)\}.$$

So far we have proved that $|a^*_{n,1}(f^*)| = O\{n^{1/4}\exp(-\tau N)\}$ for $\tau \in (0,\tau_0)$. Therefore, if $\tau < \tau+\epsilon < \tau_0$, then $|a^*_{n,1}(f^*)| = O\{n^{1/4}\exp[-(\tau+\epsilon)]N\}$ $= O\{n^{1/4}\exp(-\epsilon N)\exp(-\tau N)\}$, i.e (2.30) holds for $\tau \in (0,\tau_0)$. Further, taking into account (2.29) as well as that $a^*_n(f^*) = a^*_{n,1}(f^*) + a^*_{n,2}(f^*),\ n = 0,1,2,\dots,$ we conclude that the sequence $\{a^*_n(f^*)\}_{n=0}^{\infty}$ satisfies the inequality (2.8). Thus, the sufficiency of the condition (2.18) is proved.

As a corollary of **(V.2.6)** we can state a proposition which characterizes the growth of the functions in the space $\mathcal{E}(\tau_0)$:

(V.2.7) *The function f belongs to the space $\mathcal{E}(\tau_0), 0 < \tau_0 \le \infty$, if and only if for each $\tau \in [0,\tau_0)$ there exists a constant $B(\tau)$ such that for $z = x+iy \in \overline{S}(\tau)$,*

$$|f(z)| \le B(\tau)\exp\{x^2/2 - |x|(\tau^2-y^2)^{1/2}\}. \tag{2.32}$$

If $0 \le \tau < \infty$, then we define the function $h(\tau;x,y)$ in the strip $\overline{S}(\tau)$ by the equality

$$h(\tau;x,y) = \exp\{x^2/2 - |x|(\tau^2-y^2)^{1/2}\}. \tag{2.33}$$

By $\mathcal{H}(\tau_0),\ 0 < \tau_0 \le \infty$, we denote the $\mathbb{C}$-vector subspace of $\mathcal{H}(S(\tau_0))$ consisting of those functions f for which an inequality of the kind (2.32) holds in each closed strip $\overline{S}(\tau)$ with $0 \le \tau < \tau_0$, i.e.

$$|f(z)| = |f(x+iy)| = O(h(\tau;x,y)),\ z \in \overline{S}(\tau). \tag{2.34}$$

An immediate corollary of **(V.2.7)** is the following proposition:

(V.2.8) *The spaces $\mathcal{E}(\tau_0)$ and $\mathcal{H}(\tau_0)$ coincide for every $\tau_0 \in (0,\infty]$.*

Example. If $\operatorname{Re}\zeta < 1/2$, then the entire function $\exp(\zeta z^2)$ is in the space $H(\infty)$. Indeed, if $\tau \in [0,\infty)$, then

$$B(\tau) = \sup\{|\exp(\zeta z^2)|(h(\tau;x,y))^{-1} : z = x+iy \in \overline{S}(\tau)\} < \infty.$$

If $\zeta \ge 1/2$, then the function $\exp(\zeta z^2)$ is not in the space $\mathcal{H}(\infty)$ and, in particular, this is true for the function $\exp(z^2/2)$. Since $\lim_{\lambda\to 1/2-0}\exp(\lambda z^2)$ $= \exp(z^2/2)$ uniformly on each compact subset of $\mathbb{C}$, it follows that $\mathcal{H}(\infty)$ is not a closed subspace of $\mathcal{H}(\mathbb{C})$.

If $\operatorname{Re}\zeta < 1$, then we may come to the above conclusions without using **(V.2.7)**. To that end we need to calculate the following integrals:

$$a_{2n}(\zeta) = I_{2n}^{-1}\int_{-\infty}^{\infty}\exp[-(1-\zeta)t^2]H_{2n}(t)\,dt,\ n = 0,1,2,\dots. \tag{2.35}$$

It is easy to see that if $\operatorname{Re}\zeta < 1$, then

$$(2.36)\quad a_{2n}(\zeta) = (1-\zeta)^{-1/2} I_{2n}^{-1} \int_{-\infty}^{\infty} \exp(-t^2) H_{2n}[(1-\zeta)^{-1/2}t]\,dt,\ n = 0,1,2,\dots.$$

The right-hand sides of (2.35) and (2.36), as functions of ζ, are holomorphic in the half-plane $\operatorname{Re}\zeta < 1$. Replacing t by $(1-\xi)^{-1/2}t$ with $\xi < 1$, we conclude that these functions coincide on the ray $(-\infty, 1)$ and, hence, in the half-plane $\operatorname{Re}\zeta < 1$.

Having in mind [Chapter I, Exercise 13, (a)], we get

$$a_{2n}(\zeta) = \frac{(1-\zeta)^{-1/2}}{n!2^{2n}} \Big\{\frac{\zeta}{1-\zeta}\Big\}^n,\ n = 0,1,2,\dots$$

Therefore,

$$-\limsup_{n\to\infty} (4n+1)^{-1/2} \log|(4n/e)^n a_{2n}(\zeta) = \begin{cases} \infty & \text{if } \operatorname{Re}\zeta < 1/2; \\ 0 & \text{if } \operatorname{Re}\zeta = 1/2; \\ -\infty & \text{if } 1/2 < \operatorname{Re}\zeta < 1. \end{cases}$$

In this way we obtain the representation of the function $\exp(\zeta z^2)$, $\operatorname{Re}\zeta < 1/2$, as a series in Hermite polynomials in the whole complex plane:

$$(2.37)\qquad \exp(\zeta z^2) = \sum_{n=0}^{\infty} \frac{(1-\zeta)^{-1/2}}{n!2^{2n}} \Big\{\frac{\zeta}{1-\zeta}\Big\}^n H_{2n}(z).$$

2.5 There are cases when the expansion of a complex-valued function $f \in \mathcal{H}(S(\tau_0)), 0 < \tau_0 \le \infty$, in a series of Hermite polynomials can be obtained only by using the Christoffel-Darboux formula for the Hermite system as well as the asymptotic properties of Hermite polynomials and Hermite associated functions.

(V.2.9) *Suppose that $0 < \tau < \infty$ and let φ be a locally L-integrable complex-valued function on the line $l(-\tau) : \zeta = t - i\tau, -\infty < t < \infty$. If φ satisfies the condition*

$$(2.38)\qquad \int_{-\infty}^{\infty} |t - i\tau|^{-1} |\varphi(t - i\tau)|\,dt < \infty,$$

then for $z \in S(\tau)$ the function

$$(2.39)\qquad F(z) = \frac{1}{2\pi i} \int_{l(-\tau)} \frac{\varphi(\zeta)}{\zeta - z}\,d\zeta,\ z \in \mathbb{C} \setminus l(-\tau),$$

has a representation by a series in Hermite polynomials with coefficients

$$(2.40)\qquad A_n = \frac{1}{2\pi i I_n} \int_{l(-\tau)} G_n^-(\zeta)\varphi(\zeta)\,d\zeta,\ n = 0,1,2,\dots.$$

Proof. From (2.39) and [Chapter III, (4.14)] it follows that if $z \in \mathbb{C} \setminus l(-\tau)$ and $\nu = 0, 1, 2, \dots$, then

$$F(z) = \sum_{n=0}^{\nu} A_n H_n(z) + R_\nu(z), \tag{2.41}$$

where

$$R_\nu(z) = \frac{1}{2\pi i} \int_{l(-\tau)} \frac{\Delta_\nu(z,\zeta)\varphi(\zeta)}{\zeta - z}\, d\zeta$$

and $\Delta_\nu(z,\zeta)$ is given by [Chapter I, (4.30)]

Having in mind [Chapter III, (4.33)], the inequalities [Chapter III, (4.24)] and [Chapter III, (5.3)] as well as the Stirling formula, we obtain that for $\delta \in (0,\tau)$ and $z = x + iy \in \overline{S}(\tau - \delta)$,

$$|R_\nu(z)| = O\Big\{\exp[x^2 - \delta\sqrt{2\nu+1}] \int_{-\infty}^{\infty} |t - i\tau - z||\varphi(t - i\tau)|\, dt\Big\}.$$

But since $\sup\{|t - i\tau||t - i\tau - z|^{-1} : t \in \mathbb{R}\} < \infty$ for $z \in \overline{S}(\tau - \delta)$, from (2.38) it follows that

$$\int_{-\infty}^{\infty} |t - i\tau - z|^{-1}|\varphi(t - i\tau)|\, dt < \infty,$$

hence, $\lim_{\nu\to\infty} R_\nu(z) = 0$. Then (2.41) yields that the series in the Hermite polynomials with coefficients (2.40) is convergent in the strip $S(\tau)$ and that the function (2.39) is its sum.

Remark. It is clear that the assertion **(V.2.9)** remains true if we replace $l(-\tau)$ by the line $l(\tau) : \zeta = t + i\tau, -\infty < t < \infty$.

(V.2.10) *Suppose that the function* $f \in \mathcal{H}(S(\tau_0)), 0 < \tau_0 \le \infty$, *satisfies the following condition: for every* $\tau \in (0,\tau_0)$ *there exists* $\delta(\tau) > 0$ *such that* $|f(z)| = O(|z|^{-\delta(\tau)})$ *when* $z \to \infty$ *in* $\overline{(S)}(\tau)$. *Then* f *has an expansion for* $z \in S(\tau_0)$ *in the Hermite polynomials with coefficients* $(n = 0, 1, 2, \dots)$

$$a_n = \frac{1}{2\pi i I_n} \int_{-\infty}^{\infty} \{G_n^-(t - i\tau) f(t - i\tau) - G_n^+(t + i\tau) f(t + i\tau)\}\, dt,\ 0 < \tau < \tau_0. \tag{2.42}$$

Proof. For $z \in S(\tau_0)$ we define the functions

$$F(z) = \frac{1}{2\pi i} \int_{l(-\tau)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

and

$$G(z) = \frac{1}{2\pi i} \int_{l(\tau)} f(\zeta)\, d\zeta,$$

provided τ is chosen so that $|\operatorname{Im} z| < \tau < \tau_0$. These functions are holomorphic in $S(\tau_0)$ and, as it is easy to prove by means of the Caychy integral formula, the equality $f(z) = F(z) - G(z)$ holds for $z \in S(\tau_0)$.

Further, from **(V.2.9)** it follows that

$$F(z) = \sum_{n=0}^{\infty} A_n H_n(z)$$

and

$$G(z) = \sum_{n=0}^{\infty} B_n H_n(z)$$

for $z \in S(\tau_0)$ and, moreover,

$$A_n = \frac{1}{2\pi i I_n} \int_{l(-\tau)} G_n^-(\zeta) f(\zeta)\, d\zeta = \frac{1}{2\pi i I_n} \int_{-\infty}^{\infty} G_n^-(t - i\tau) f(t - i\tau)\, dt$$

and

$$B_n = \frac{1}{2\pi i I_n} \int_{l(\tau)} G_n^+(\zeta) f(\zeta)\, d\zeta = \frac{1}{2\pi i I_n} \int_{-\infty}^{\infty} G_n^+(t + i\tau) f(t + i\tau)\, dt.$$

for $n = 0, 1, 2, \ldots$.

Thus, we come to the representation of the function f in $S(\tau_0)$ by a series in the Hermite polynomials with coefficients given by the equalities $a_n = A_n - B_n$, $n = 0, 1, 2, \ldots$.

Remark. The fact that a function $f \in \mathcal{H}(S(\tau_0))$ satisfying the hypothesis of **(V.2.10)** has a representation by a series in the Hermite polynomials in the strip $S(\tau_0)$ follows immediately from **(V.2.7)**. The essential thing is that the coefficients $\{a_n\}_{n=0}^{\infty}$ have representations in terms of the Hermite associated functions by means of the equalities (2.42).

3. Expansions in series of Laguerre polynomials

3.1 Suppose that $\alpha \in \mathbb{C}, 0 < \lambda \leq \infty$, and denote by $\mathcal{L}^{(\alpha)}(\lambda_0)$ the $\mathbb{C}$-vector space of the complex-valued functions which are holomorphic in the region $\Delta(\lambda_0)$, and which are representable there by series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$. It is not difficult to prove that $\mathcal{L}^{(\alpha)}(\lambda_0)$ is a proper $\mathbb{C}$-vector subspace of $\mathcal{H}(\Delta(\lambda_0))$. Indeed, the following proposition is true:

(V.3.1) *Suppose that $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$. If the complex function $f \in \mathcal{H}(\Delta(\lambda_0))$ is representable for $z \in \Delta(\lambda_0)$ by a series in the polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$, then for $\lambda \in (0, \lambda_0)$ and $\rho > \max(1, 2\lambda)$,*

$$|f(z)| = |f(x + iy)| = O(|z|^{-\alpha/2-1/4} \exp x) \tag{3.1}$$

on the closed region $\tilde{\Delta}(\lambda,\rho)=\overline{\Delta(\lambda)}\cap\{z\in\mathbb{C}:|z|\geq\rho\}$.

Proof. If we assume that the representation [Chapter IV, (2.1)] holds for f in the region $\Delta(\lambda_0)$, then from **(IV.2.1)** it follows that $|a_n|=O(\exp[-2(\lambda+\delta)\sqrt{n}])$ provided $0<\lambda<\lambda+\delta<\lambda_0$. Further, from [Chapter III, (4.21)] we find that if $z\in\tilde{\Delta}(\lambda,\rho)$, then

$$|f(z)|\leq\sum_{n=0}^{\infty}|a_nL_n^{(\alpha)}(z)|$$

$$\leq|a_0||L_0^{(\alpha)}(z)|+O\left(|z|^{-\alpha/2-1/4}\exp x\cdot\sum_{n=1}^{\infty}n^{\alpha/2-1/4}\exp(-2\delta\sqrt{n})\right),$$

and, since $L_0^{(\alpha)}(z)$ is a constant, we obtain the inequality (3.1).

3.2 A growth characteristic of the functions in the space $\mathcal{L}^{(\alpha)}(\lambda_0)$ can be obtained by the aid of the proposition **(V.2.7)** which itself is a corollary of **(V.2.6)**. Recall that by $\mathcal{E}(\tau_0), 0<\tau_0\leq\infty$, we denoted the $\mathbb{C}$-vector space of the complex-valued functions holomorphic in the strip $S(\tau_0)$ and having there expansions in Hermite polynomials. The following proposition plays an essential role in the further considerations.

(V.3.2) *If* $f\in\mathcal{E}(\tau_0), 0<\tau_0\leq\infty$, *and* φ *is a L-integrable complex function on the interval* $[0,1]$, *then*:

(a) $zf(z)\in\mathcal{E}(\tau_0)$;

(b) $\displaystyle\int_0^1\varphi(t)f(zt)\,dt\in\mathcal{E}(\tau_0)$;

(c) $f'(z)\in\mathcal{E}(\tau_0)$.

Proof. **(a)** If $0\leq\tau<\tau+\delta<\tau_0$, then for $z=x+iy\in\overline{S}(\tau)$ we have

$$|zf(z)|=O\{(x^2+y^2)^{1/2}h(\tau+\delta;x,y)\}$$

$$=O\{h(\tau;x,y)(x^2+\tau^2)^{1/2}h(\tau+\delta;x,y)[h(\tau;x,y)^]-1\}$$

Since $[(\tau+\delta)^2-y^2]^{1/2}-(\tau^2-y^2)^{1/2}\geq\delta$ when $|y|\leq\tau$, it follows that

$$(x^2+\tau^2)h(\tau+\delta;x,y)[h(\tau;x,y)]^{-1}$$

$$=(x^2+\tau^2)^{1/2}\exp\{-|x|[(\tau+\delta)^2-y^2]^{1/2}-(\tau^2-y^2)^{1/2}]\}$$

$$\leq(x^2+\tau^2)^{1/2}\exp(\delta|x|)=O(1).$$

Hence, $|zf(z)|=O(h(\tau;x,y))$ when $-\infty<x<\infty$ and $|y|\leq\tau$, i.e $zf(z)\in\mathcal{H}(\tau_0)$ and then **(V.2.8)** yields that $zf(z)\in\mathcal{E}(\tau_0)$.

(b) Suppose that $0<\tau<\tau_0$. If $t\in[0,1]$ is fixed, then

$$\theta(t,u)=t(1-t^2u)^{1/2}-(1-u)^{1/2}, 0\leq u\leq 1,$$

as a function of u, is not decreasing since $\theta'_u(t,u) \geq 0$ when $0 \leq u < 1$. Therefore, $\theta(t,u) \geq \theta(t,0)$, i.e. $t(1-t^2u)^{1/2} - (1-u)^{1/2} \geq -(1-t), 0 \leq u \leq 1$. Setting $u = y^2/\tau^2$, we obtain the inequality $t(\tau^2 - t^2y^2)^{1/2} - (\tau^2 - y^2)^{1/2} \geq -\tau(1-t)$ which is valid when $|y| \leq \tau$. Hence, if $|x| \geq 2\tau$ and $0 \leq t \leq 1$, then

$$\begin{aligned} &h(\tau; tx, ty)[h(\tau; x, y)]^{-1} \\ &= \exp\{-(1-t^2)x^2/2 - |x|[t(\tau^2 - t^2y^2)^{1/2} - (\tau^2 - y^2)^{1/2}]\} \\ &\leq \exp\{-(1-t)[(1+t)x^2/2 - \tau|x|]\} \leq 1. \end{aligned}$$

If $z = x + iy \in \overline{S}(\tau)$, then

$$\begin{aligned} \Big|\int_0^1 \varphi(t) f(zt)\, dt\Big| &\leq \max\{|f(zt)| : 0 \leq t \leq 1\} \int_0^1 |\varphi(t)|\, dt \\ &= O\{\max[h(\tau; tx, ty) : 0 \leq t \leq 1]\} = O\{h(\tau; x, y)\}. \end{aligned}$$

(c) Suppose that the series representation [Chapter IV, (54.27)] holds for a function $f \in \mathcal{H}(S(\tau_0))$. Then from the uniform convergence of this series on the compact subsets of the strip $S(\tau_0)$, and from the equalities $H'_n(z) = 2nH_{n-1}(z)$, $n = 1, 2, 3, \dots$ [Chapter I, Exercise 11] it follows that $f'(z) = \sum_{n=0}^{\infty} 2(n+1)a_{n+1}H_n(z)$ in $S(\tau_0)$, i.e. $f' \in \mathcal{E}(\tau_0)$.

(V.3.3) *If $-1/2 < \operatorname{Re}\alpha < 1/2$, then the integral transform*

$$f(z) = P^{(\alpha)}(F; z) = \frac{1}{\Gamma(\alpha + 1/2)} \int_0^1 (1-t^2)^{\alpha - 1/2} F(zt)\, dt \tag{3.2}$$

is an automorphism of the space $\mathcal{E}(\tau_0)$.

Proof. From **(V.3.2),(b)** it follows that if $F \in \mathcal{E}(\tau_0)$, then $f \in \mathcal{E}(\tau_0)$. The equalities

$$f^{(k)}(0) = \frac{\Gamma(k/2 + 1/2)}{2\Gamma(\alpha + k/2 + 1/2)} F^{(k)}(0),\ k = 0, 1, 2, \dots$$

imply immediately that the mapping $P^{(\alpha)}$ is injective.

Define

$$F(z) = Q^{(\alpha)}(f; z) = \frac{d}{dz}\left\{\frac{2z}{\Gamma(-\alpha/2 + 1/2)} \int_0^1 (1-t^2)^{-\alpha/2 - 1/2} t^{2\alpha+1} f(zt)\, dt\right\}. \tag{3.3}$$

From **(V.3.2)** it follows that if $f \in \mathcal{E}(\tau_0)$, then $F \in \mathcal{E}(\tau_0)$. Using power series representation for the functions F and f centered at the origin we easily prove that $P^{(\alpha)}(F; z) = f(z)$ for $z \in U(0; \tau_0)$ and, hence, in the whole strip $S(\tau_0)$, i.e. the mapping $P^{(\alpha)}$ is surjective.

Suppose that the sequence $\{F_n(z)\}_{n=1}^{\infty} \subset \mathcal{H}(S(\tau_0))$ is convergent in the space $\mathcal{H}(S(\tau_0))$ to the function $F \in \mathcal{H}(S(\tau_0))$. If $T > 0$ and $0 \le \tau < \tau_0$, then it is uniformly convergent on the compact set $K(T,\tau) = \{z = x+iy : |x| \le T, |y| \le \tau\}$. From (3.2) it follows that the sequence $\{P^{(\alpha)}(F_n; z)\}_{n=1}^{\infty}$ is uniformly convergent on the set $K(T,\tau)$ to the function $P^{(\alpha)}(F;z)$. Since each compact subset of $S(\tau_0)$ is contained in a set of the kind $K(T,\tau)$, we find that $\lim_{n\to\infty} P^{(\alpha)}(F_n;z)$ $= P^{(\alpha)}(F;z)$ with respect to the topology of the space $\mathcal{H}(S(\tau_0))$. Since this topology is metrizible, it follows that the mapping $P^{(\alpha)}$ is continuous. In a similar way we prove that the mapping $Q^{(\alpha)}$ is also continuous. Since we consider $\mathcal{E}(\tau_0)$ as a topological vector space with respect to the topology induced by that of $\mathcal{H}(S(\tau_0))$, we obtain that $P^{(\alpha)}$ and $Q^{(\alpha)}$ are continuous as mappings of the space $\mathcal{E}(\tau_0)$.

Remark. The integral transform $P^{(\alpha)}$ is, in fact, an operator of fractional differentiation of Riemann-Liuville's type and $Q^{(\alpha)}$ is its inverse.

Denote by $\tilde{\mathcal{E}}(\tau_0), 0 < \tau_0 \le \infty$, the set of even functions in $\mathcal{E}(\tau_0)$, i.e. $\tilde{\mathcal{E}}(\tau_0)$ is the space of all complex functions which are holomorphic in the region $S(\tau_0)$ and which have there expansions in series of the polynomials $\{H_{2n}(z)\}_{n=0}^{\infty}$. Evidently, the restriction of the operator $P^{(\alpha)}$ to the subspace $\tilde{\mathcal{E}}(\tau_0)$ is an automorphism of this subspace. Moreover, since $\{H_{2n}(z)\}_{n=0}^{\infty}$ is a basis of $\tilde{\mathcal{E}}(\tau_0)$, as an immediate corollary of **(V.3.3)** and **(II.2.7)** we obtain the following proposition:

(V.3.4) *If* $\tau_0 \in (0,\infty]$ *and* $-1/2 < \operatorname{Re}\alpha < 1/2$, *then the system of polynomials* $\{L_n^{(\alpha)}(z^2)\}_{n=0}^{\infty}$ *is a basis of the space* $\tilde{\mathcal{E}}(\tau_0)$.

Now, we are able to give a growth characteristic of the functions in the space $\mathcal{L}^{(\alpha)}(\lambda_0)$, i.e. of the complex functions which are holomorphic in a region of the kind $\Delta(\lambda_0), 0 < \lambda_0 \le \infty$, and have there representations as series in Laguerre polynomials with parameter $\alpha \in \mathbb{C}$. Moreover, we are to make clear for which α the system of polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ is a basis in the space $\mathcal{L}^{(\alpha)}(\lambda_0)$.

(V.3.5) *Suppose that* $0 < \lambda_0 \le \infty$ *and* $\alpha \in \mathbb{C}$. *Then* $\mathcal{L}^{(\alpha)}(\lambda_0) \subset \mathcal{L}^{(\alpha+1)}(\lambda_0)$. *If* $f(z) \in \mathcal{L}^{(\alpha+1)}(\lambda_0)$, *then* $zf(z) \in \mathcal{L}^{(\alpha)}(\lambda_0)$.

Proof. Suppose that the representation [Chapter IV, (4.6)] holds for the function $f \in \mathcal{H}(\Delta(\lambda_0))$ in the region $\Delta(\lambda_0)$. Then, having in mind the relation [Chapter III, (2.10)], we find that

$$f(z) = \sum_{n=0}^{\infty}(a_n - a_{n+1})L_n^{(\alpha+1)}(z),\ z \in \Delta(\lambda_0), \tag{3.4}$$

i.e. $f \in \mathcal{L}^{(\alpha+1)}(\lambda_0)$.

Conversely, if $f(z) \in \mathcal{L}^{(\alpha+1)}(\lambda_0)$, then from the relation [Chapter I, Exercise 6, (b)], i.e.

$$zL_n^{(\alpha+1)}(z) = (n+\alpha+1)L_n^{(\alpha)}(z) - (n+1)L_{n+1}^{(\alpha)}(z),\ n = 0,1,2,\ldots, \tag{3.5}$$

it follows that the function $zf(z)$ is representable by a series in the polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ in the region $\Delta(\lambda_0)$. Indeed, **(IV.2.1),(b)** and the equality

$$-\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|na_n| = -\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n| \tag{3.6}$$

yield that if the series $\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z)$ converges in the region $\Delta(\lambda_0)$, then so does the series $\sum_{n=0}^{\infty} na_n L_n^{(\alpha)}(z)$.

If $0<\lambda<\infty$, then we denoted by $\overline{\Delta}(\lambda)$ the closure of the region $\overline{\Delta(\lambda)}$, i.e. $\overline{\Delta}(\lambda) = \{z\in\mathbb{C} : \mathrm{Re}(-z)^{1/2}\le\lambda)\}$. Define, in addition, $\overline{\Delta}(0)=[0,\infty)$. Further, by $\varphi(\lambda;x,y)$ we denote the function defined in $\overline{\Delta}(\lambda)$ as follows:

$$\varphi(\lambda;x,y)$$

$$= \exp\left\{\frac{\sqrt{x^2+y^2}+x}{4} - \left[\frac{\sqrt{x^2-y^2}+x}{2}\left(\lambda^2 - \frac{\sqrt{x^2+y^2}-x}{2}\right)\right]^{1/2}\right\}. \tag{3.7}$$

The equality $z=\zeta^2$ with $z=x+iy$ and $\zeta=\xi+i\eta$ gives that $\xi^2 = (\sqrt{x^2+y^2} +x)/2$ and $\eta^2 = (\sqrt{x^2+y^2}-x)/2$. Moreover, if $\zeta\in\overline{S}(\lambda)=\{w\in\mathbb{C} : |\mathrm{Im}\,w| \le\lambda\}, 0\le\lambda<\infty$, then $z\in\overline{\Delta}(\lambda)$ and (2.34) yields that

$$h(\lambda;\xi,\eta)=\varphi(\lambda;x,y). \tag{3.8}$$

Denote by $\mathcal{L}(\lambda_0), 0<\lambda_0\le\infty$, the $\mathbb{C}$-vector subspace of $\mathcal{H}(\Delta(\lambda_0))$ consisting of the functions f which satisfy the following condition: for $\lambda\in[0,\lambda_0)$ there exists a constant $D=D(\lambda)\ge 0$ such that the inequality

$$|f(z)| = |f(x+iy)| \le D(\lambda)\varphi(\lambda;x,y). \tag{3.9}$$

holds for $z=x+iy\in\overline{\Delta}(\lambda)$.

Further, denote $A_k=\{z\in\mathbb{C} : \mathrm{Re}\,z = k+1/2, z\ne k+1/2\}, k=0,\pm1,\pm2,\dots$ and define $\mathbb{A}=\bigcup_{k\in\mathbb{Z}} A_k$.

(V.3.6) *If $\alpha\in\mathbb{C}\setminus\mathbb{A}$ and $\lambda_0\in(0,\infty]$, then the space $\mathcal{L}^{(\alpha)}(\lambda_0)$ coincides with the space $\mathcal{L}(\lambda_0)$.*

Proof. Suppose that $\mathcal{L}^{(\alpha)}(\lambda_0)=\mathcal{L}(\lambda_0)$ for some $\alpha\in\mathbb{C}$. If the function $f \in\mathcal{L}^{(\alpha+1)}(\lambda_0)$, then from **(V.3.5)** it follows that $zf(z)\in\mathcal{L}^{(\alpha)}(\lambda_0)$, i.e. $zf(z)\in \mathcal{L}(\lambda_0)$, hence, $f(z)\in\mathcal{L}(\lambda_0)$. If, conversely, $f\in\mathcal{L}(\lambda_0)$, then again **(V.3.6)** yields that $f\in\mathcal{L}^{(\alpha+1)}(\lambda_0)$. In other words, if the proposition which we wish to prove holds for some $\alpha\in\mathbb{C}$, then it holds for $\alpha+1$ too.

Suppose now that $\mathcal{L}^{(\alpha+1)}(\lambda_0) = \mathcal{L}(\lambda_0)$ with $\alpha \in \mathbb{C}$. If $f \in \mathcal{L}^{(\alpha)}(\lambda_0)$, then $f \in \mathcal{L}^{(\alpha+1)}(\lambda_0)$, hence, $f \in \mathcal{L}(\lambda_0)$. If, conversely, $f \in \mathcal{L}(\lambda_0)$, then the function $z^{-1}\{f(z) - f(0)\} \in \mathcal{L}(\lambda_0) = \mathcal{L}^{(\alpha+1)}(\lambda_0)$ and, using again **(V.3.6)**, we find that $f(z) - f(0) \in \mathcal{L}^{(\alpha)}(\lambda_0)$, i.e. $f \in \mathcal{L}^{(\alpha)}(\lambda_0)$.

From this considerations it follows that it is sufficient to prove the validity of **(V.3.6)** if $-1/2 < \operatorname{Re}\alpha < 1/2$ as well as if $\alpha = -1/2$.

Suppose that $-1/2 < \operatorname{Re}\alpha < 1/2$ and let $f \in \mathcal{L}^{(\alpha)}(\lambda_0)$, i.e. the function $f \in \mathcal{H}(\Delta(\lambda_0))$ is representable by a series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ in the region $\Delta(\lambda_0)$. Then the function $f(\zeta^2) \in \mathcal{H}(S(\lambda_0))$ has an expansion in a series of the polynomials $\{L_n^{(\alpha)}(\zeta^2)\}_{n=0}^{\infty}$ in the region $S(\lambda_0)$. Then from **(V.3.4)** it follows that for $\lambda \in [0, \lambda_0)$, $f(\zeta^2) \in \tilde{\mathcal{E}}(\lambda) \subset \mathcal{E}(\lambda)$, and from **(V.2.7)** we find that $|f(\zeta^2)| = O(h(\lambda; \xi, \eta))$ for $\zeta = \xi + i\eta \in \overline{S}(\lambda)$. Finally, (3.8) yields that $|f(z)| = O(\varphi(\lambda; x, y))$ for $z = x + iy \in \overline{\Delta}(\lambda)$, i.e. $f \in \mathcal{L}(\lambda_0)$.

If, conversely, $f \in \mathcal{L}(\lambda_0)$, then for $\lambda \in [0, \lambda_0)$ we have $|f(\zeta^2)| = O(h(\lambda; \xi, \eta))$ for $\zeta = \xi + i\eta \in \overline{S}(\lambda)$ and from **(V.2.7)** we obtain that $f(\zeta^2) \in \tilde{\mathcal{E}}(\lambda_0)$. It means, due to **(V.3.4)**, that the function $f(\zeta^2)$ is representable in the strip $S(\lambda_0)$ by a series in the polynomials $\{L_n^{(\alpha)}(\zeta^2)\}_{n=0}^{\infty}$, i.e. the function $f(z)$ has an expansion in a series of the polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ in the region $\Delta(\lambda_0)$, hence, $f \in \mathcal{L}^{(\alpha)}(\lambda_0)$.

It remains to consider the case $\alpha = -1/2$. Suppose that $f \in \mathcal{L}^{(-1/2)}(\lambda_0)$. If we take into account the relation [Chapter I, Exercise 14,(a)], i.e.

$$(3.10) \qquad L_n^{(-1/2)}(\zeta^2) = (-1)^n 2^{-2n} (n!)^{-1} H_{2n}(\zeta), \ n = 0, 1, 2, \ldots,$$

then we can conclude that the function $f(\zeta^2) \in \mathcal{H}(S(\lambda_0))$ is representable by a series in the polynomials $\{H_{2n}(\zeta)\}_{n=0}^{\infty}$ in the strip $S(\lambda_0)$, i.e. it belongs to the space $\tilde{\mathcal{E}}(\lambda_0)$. Then from **(V.2.7)** it follows that for $\lambda \in [0, \lambda_0)$, $|f(\zeta^2)| = O(h(\lambda; \xi, \eta))$ for $\zeta = \xi + i\eta \in \overline{S}(\lambda)$ and, hence, $|f(z)| = O(\varphi(\lambda; x, y))$ for $z = x + iy \in \overline{\Delta}(\lambda)$, i.e. $f \in \mathcal{L}(\lambda_0)$.

Let, conversely, $f \in \mathcal{L}(\lambda_0)$. Then for the even function $f(\zeta^2)$ we have the estimate $|f(\zeta^2)| = O(h(\lambda; \xi, \eta))$ for $\zeta = \xi + i\eta \in \overline{S}(\lambda), 0 \le \lambda < \lambda_0$. Further, from **(V.2.7)** it follows that this function is representable by a series in the polynomials $\{H_{2n}(z)\}_{n=0}^{\infty}$ in the region $S(\lambda_0)$, and from relation (3.10) we obtain immediately that the function $f(\zeta^2)$ has an expansion in a series of the polynomials $\{L_n^{(-1/2)}(\zeta^2)\}_{n=0}^{\infty}$ in the strip $S(\lambda_0)$. Hence, the function $f(z)$ is representable by a series in the polynomials $\{L_n^{(-1/2)}(z)\}_{n=0}^{\infty}$ in the region $\Delta(\lambda_0)$, i.e. $f \in \mathcal{L}^{(-1/2)}(\lambda_0)$.

Denote by $L^{(\alpha)}$ the system of the Laguerre polynomials with parameter α, i.e. $L^{(\alpha)} = \{L_n^{(\alpha)}(z)\}_{n=0}^{\alpha}$.

(V.3.7) *If $\alpha \in \mathbb{C} \setminus (\mathbb{A} \cup \mathbb{Z}^-)$ and $0 < \lambda_0 \le \infty$, then the system $L^{(\alpha)}$ is a basis of the space $\mathcal{L}^{(\alpha)}(\lambda_0)$.*

Proof. Suppose that the system $L^{(\alpha)}$ is a basis of $\mathcal{L}^{(\alpha)}(\lambda_0)$. If the series

$\sum_{n=0}^{\infty} a_n^{(\alpha+1)} L_n^{(\alpha+1)}(z)$ is convergent in the region $\Delta(\lambda_0)$ and its sum is identically zero, then the same holds for the series $\sum_{n=0}^{\infty} a_n^{(\alpha+1)} z L_n^{(\alpha+1)}(z)$. Then from relation (3.6) we obtain that

$$(\alpha+1)a_0^{(\alpha+1)} L_0^{(\alpha)}(z) + \sum_{n=1}^{\infty}\{(n+\alpha+1)a_n^{(\alpha+1)} - n a_{n-1}^{(\alpha+1)}\} L_n^{(\alpha)}(z) = 0$$

for $z \in \Delta(\lambda_0)$. Therefore, $(\alpha+1)a_0^{(\alpha+1)} = 0$ and $(n+\alpha+1)a_n^{(\alpha+1)} - a_{n-1}^{(\alpha+1)} = 0$ for $n = 1, 2, 3, \dots$ and since $\alpha + 1 \neq 0, -1, -2, \dots$, we find that $a_n^{(\alpha+1)} = 0$ for $n = 0, 1, 2, \dots$.

Suppose now that $L^{(\alpha+1)}$ is a basis in $\mathcal{L}^{(\alpha+1)}(\lambda_0)$. If the series $\sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(z)$ is convergent in $\Delta(\lambda_0)$, and its sum is identically zero, then the relations (3.5) give $\sum_{n=0}^{\infty}(a_n^{(\alpha)} - a_{n+1}^{(\alpha)}) L_n^{(\alpha+1)}(z) = 0$ for $z \in \Delta(\lambda_0)$. Hence, $a_n^{(\alpha)} - a_{n+1}^{(\alpha)} = 0$, $n = 0, 1, 2, \dots$, i.e. $a_n^{(\alpha)} = a_0^{(\alpha)}, n = 1, 2, 3, \dots$. But the series $\sum_{n=0}^{\infty} a_0^{(\alpha)} L_n^{(\alpha)}(z)$ is convergent in a region of the kind $\Delta(\lambda_0)$ with $0 < \lambda_0 \le \infty$ if and only if $a_0^{(\alpha)} = 0$.

From the above considerations it follows that it is sufficient to prove the assertion only when $-1/2 < \alpha < 1/2$ as well as in the case $\alpha = -1/2$. If $-1/2 < \alpha < 1/2$, then the system $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ is a basis of the space $\mathcal{L}^{(\alpha)}(\lambda_0)$ since, due to **(V.3.5)**, the system $\{L_n^{(\alpha)}(z^2)\}_{n=0}^{\infty}$ is a basis of the space $\tilde{\mathcal{E}}(\lambda_0)$. The system $\{L_n^{(-1/2)}(z)\}_{n=0}^{\infty}$ is a basis in $\mathcal{L}^{(-1/2)}(\lambda_0)$ because of the relation (3.10) and the fact that the system $\{H_{2n}(z)\}_{n=0}^{\infty}$ is a basis in $\tilde{\mathcal{E}}(\lambda_0)$.

3.3 Now, as an application of **(V.3.7)**, we are going to prove a proposition which could be considered as a generalization of **(IV.5.2)**.

(V.3.8) *Suppose that $0 < \lambda_0 \le \infty, \alpha \in \mathbb{R} \setminus \mathbb{Z}^-$ and $f \in \mathcal{L}^{(\alpha)}(\lambda_0)$. Denote by $\{a_n^{(\alpha)}(f)\}_{n=0}^{\infty}$ the coefficients of the series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ representing the function f in the region $\Delta(\lambda_0)$ and let $0 < \lambda < \lambda_0$.*

(a) *If α is not an integer, then*

$$(3.11) \quad a_n^{(\alpha)}(f) = \frac{1}{2iI_n^{(\alpha)} \sin\alpha\pi} \int_{p(\lambda)} (-z)^{\alpha} \exp(-z) L_n^{(\alpha)}(z) f(z)\, dz \;\; n = 0, 1, 2, \dots;$$

(b) *for $k = 0, 1, 2, \dots$,*

$$(3.12) \quad a_n^{(k)} = \frac{(-1)^k}{2\pi i I_n^{(k)}} \int_{p(\lambda)} (-z)^k \exp(-z) \log(-z) L_n^{(k)}(z) f(z)\, dz, \;\; n = 0, 1, 2, \dots.$$

Proof. Since $\mathcal{L}^{(\alpha)}(\lambda_0) = \mathcal{L}(\lambda_0)$, we have $|f(z)| = O(\varphi(\lambda; x, y))$ for $\lambda \in (0, \lambda_0)$ and for $z = x + iy \in \overline{\Delta}(\lambda)$. In particular, $|f(z)| = O(\exp(x/2))$ on the parabola $p(\lambda) = \partial\overline{\Delta}(\lambda)$. Therefore, the integrals on the right-hand sides of (3.11) and (3.12) are absolutely convergent for $n = 0, 1, 2, \dots$ provided $0 < \lambda < \lambda_0$.

From [III, (2.10)] it follows that $a_n^{(\alpha+1)} = a_n^{(\alpha)} - a_{n+1}^{(\alpha)}$, $n = 0, 1, 2, \dots$. Moreover, the equalities [IV, (5.8)] give that $f'(z) = -\sum_{n=0}^{\infty} a_{n+1}^{(\alpha)}(f) L_n^{(\alpha+1)}(z)$ for $z \in \Delta(\lambda_0)$ and, hence, $a_n^{(\alpha+1)}(f') = -a_{n+1}^{(\alpha)}(f)$, $n = 0, 1, 2, \dots$.

Suppose that the part **(a)** of the proposition holds if $\alpha + 1 \in \mathbb{R} \setminus \mathbb{Z}$. Integrating by parts we obtain

$$2iI_0^{(\alpha+1)} \sin(\alpha+1)\pi . a_0^{(\alpha+1)}(f)$$
$$= \int_{p(\lambda)} [-(\alpha+1)(-z)^{\alpha} f(z) + (-z)^{\alpha+1} f'(z)] \exp(-z)\, dz.$$

Since $I_0^{(\alpha+1)} = (\alpha+1) I_0^{(\alpha)}$ and $\sin(\alpha+1)\pi = -\sin\alpha\pi$, we find that

$$a_0^{(\alpha+1)} = \frac{1}{2iI_0^{(\alpha)} \sin\alpha\pi} \int_{p(\lambda)} (-z)^{\alpha} \exp(-z) f(z)\, dz + a_0^{(\alpha+1)}(f').$$

Since $a_0^{(\alpha)}(f) = a_0^{(\alpha+1)}(f) + a_1^{(\alpha)}(f) = a_0^{(\alpha+1)}(f) - a_0^{(\alpha+1)}(f')$, we conclude that

$$a_0^{(\alpha)}(f) = \frac{1}{2iI_0^{(\alpha)} \sin\alpha\pi} \int_{p(\lambda)} (-z)^{\alpha} \exp(-z) f(z)\, dz.$$

The above reasoning shows that if **(a)** holds for $\alpha + 1$ and $n = 0$, then it holds for α and $n = 0$ too.

Suppose now that **(a)** holds for $\alpha + 1$ and for some $n \geq 1$. Then, having in mind (3.6) as well as the relations $(n + \alpha + 1)/I_n^{(\alpha+1)} = 1/I_n^{(\alpha)}, (n+1)/I_n^{(\alpha+1)} = 1/I_{n+1}^{(\alpha)}$, $n = 0, 1, 2, \dots$, we find that

$$a_n^{(\alpha+1)}(f) = \frac{1}{2iI_n^{\alpha+1} \sin(\alpha+1)\pi} \int_{p(\lambda)} (-z)^{(\alpha+1)} \exp(-z) L_n^{(\alpha+1)}(z) f(z)\, dz$$

$$= \frac{1}{2iI_n^{(\alpha+1)} \sin\alpha\pi} \int_{p(\lambda)} (-z)^{\alpha} \exp(-z)[(n+\alpha+1)L_n^{(\alpha)}(z) - (n+1)L_{n+1}^{(\alpha)}(z)] f(z)\, dz$$
$$= a_n^{(\alpha)}(f) - \frac{1}{2iI_{n+1}^{(\alpha)} \sin\alpha\pi} \int_{p(\lambda)} (-z)^{\alpha} \exp(-z) L_{n+1}^{(\alpha)}(z) f(z)\, dz.$$

Since $a_{n+1}^{(\alpha)}(f) = a_n^{(\alpha)}(f) - a_n^{(\alpha+1)}(f)$, we get finally

$$a_{n+1}^{(\alpha)}(f) = \frac{1}{2iI_{n+1} \sin\alpha\pi} \int_{p(\lambda)} (-z)^{\alpha} \exp(-z) L_{n+1}^{(\alpha)}(z) f(z)\, dz.$$

It means that if the part **(a)** holds for $\alpha + 1 \in \mathbb{R} \setminus \mathbb{Z}$, then it holds for α too, i.e. it is sufficient to prove it when $\alpha > -1$.

As in the proof of **(II.2.4)**, denote by $R \pm i\sigma(\lambda, R)$ the points of intersection of the parabola $p(\lambda)$ and the line $\operatorname{Re} z = R > 0$. Since $|f(z)| = O(\exp(R/2))$ for $z = R + iy \in \Delta(\lambda), 0 < \lambda < \lambda_0$, the Cauchy integral theorem yields that the integrals in (3.11) and (3.12) in fact does not depend on $\lambda \in (0, \lambda_0)$. But if $\alpha > -1$ is not an integer, then

$$\frac{1}{2iI_n^{(\alpha)} \sin \alpha\pi} \lim_{\lambda \to 0} \int_{p(\lambda)} (-z)^\alpha \exp(-z) L_n^{(\alpha)}(z) f(z)\, dz$$

$$= \frac{1}{I_n^{(\alpha)}} \int_0^\infty t^\alpha \exp(-t) L_n^{(\alpha)}(t) f(t)\, dt, \ n = 0, 1, 2, \dots,$$

and the validity of the equalities (3.11) follows from **(IV.5.7)**.

If $\alpha = k$ is a nonnegative integer, then

$$\frac{(-1)^k}{2\pi i I_n^{(\alpha)}} \lim_{\lambda \to 0} \int_{p(\lambda)} (-z)^k \exp(-z) \log(-z) L_n^{(k)}(z) f(z)\, dz$$

$$= \frac{1}{I_n^{(k)}} \int_0^\infty t^k \exp(-t) L_n^{(k)}(t) f(t)\, dt, \ n = 0, 1, 2, \dots,$$

and, thus, we arrive to (3.12).

Remark. If $f(z) \equiv 1$, then $a_0^{(\alpha)} = 1$ and (3.11) yields that for $\lambda \in (0, \infty)$ and $\alpha \in \mathbb{R} \setminus \mathbb{Z}$,

$$\int_{p(\lambda)} (-z)^\alpha \exp(-z)\, dz = 2i\Gamma(\alpha + 1) \sin \alpha\pi, \tag{3.13}$$

and

$$\int_{p(\lambda)} z^k \exp(-z) \log(-z)\, dz = 2\pi i.k!. \tag{3.14}$$

for $k = 0, 1, 2, \dots$

Let us consider as an example the entire function $\exp(\zeta z)$. If $\operatorname{Re} \zeta < 1/2$, then it is in the space $\mathcal{L}(\infty)$ since $\exp(\zeta z^2) \in \mathcal{H}(\infty)$. Therefore, for each $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$ it has a representation by a series in the polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^\infty$ in the whole complex plane. Denote by $\{a_n^{(\alpha)}(\zeta)\}_{n=0}^\infty$ the coefficients of this representation. Then the Rodrigues formula for the Laguerre polynomials in the region $\mathbb{C} \setminus [0, \infty)$ [I, Exercise. 5, (c)] as well as the representations (3.11) give that if $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, then $(n = 0, 1, 2, \dots)$,

$$a_n^{(\alpha)}(\zeta) = \frac{(-1)^n}{2in! I_n^{(\alpha)} \sin \alpha\pi} \int_{p(\lambda)} \exp(\zeta z) \{(-z)^{n+\alpha} \exp(-z)\}^{(n)}\, dz.$$

Integrating by parts we obtain

$$a_n^{(\alpha)}(\zeta) = \frac{\zeta^n}{2i\Gamma(n+\alpha+1)\sin\alpha\pi}\int_{p(\lambda)}(-z)^{n+\alpha}\exp[-(1-\zeta)z]\,dz.$$

If $\zeta = \xi$ is real, i.e. $\xi \in (-\infty, 1/2)$, then

$$a_n^{(\alpha)}(\xi) = \frac{\xi^n/(1-\xi)^n}{2i\Gamma(n+\alpha+1)\sin\alpha\pi}\int_{p(\lambda)}[-z(1-\xi)]^{n+\alpha}\exp[-(1-\xi)z]\,dz.$$

By substituting $(1-\xi)^{-1}z$ for z, the parabola $p(\lambda)$ is transformed into the parabola $p((1-\xi)^{1/2}\lambda)$ and, therefore,

$$a_n^{(\alpha)}(\zeta) = \frac{\xi^n/(1-\xi)^{n+\alpha+1}}{2i\Gamma(n+\alpha+1)\sin\alpha\pi}\int_{p((1-\xi)^{1/2}\lambda)}(-z)^{n+\alpha}\exp(-z)\,dz.$$

Since $n+\alpha \in \mathbb{R}\setminus\mathbb{Z}$, (3.13) gives that

$$a_n^{(\alpha)}(\xi) = (1-\xi)^{-1-\alpha}(-\xi/(1-\xi))^n,\ n = 0,1,2,\dots \tag{3.15}$$

For $n = 0,1,3,\dots$, $a_n^{(\alpha)}(\zeta)$ and $(1-\zeta)^{-1-\alpha}(-\zeta/(1-\zeta))^n$ are holomorphic functions of ζ in the half-plane $\operatorname{Re}\zeta < 1/2$. From (3.15) and the identity theorem it follows that they coincide in this half-plane, i.e.

$$a_n^{(\alpha)}(\zeta) = (1-\zeta)^{-1-\alpha}(-\zeta/(1-\zeta))^n,\ n = 0,1,2,\dots.$$

Hence, we have the representation

$$(1-\zeta)^{1+\alpha}\exp(\zeta z) = \sum_{n=0}^{\infty}(-\zeta/(1-\zeta))^n L_n^{(\alpha)}(z),\ \operatorname{Re}\zeta < 1/2,\ z \in \mathbb{C}. \tag{3.16}$$

Its validity was proved under the assumption that $\alpha \in \mathbb{R}\setminus\mathbb{Z}$, but it remains true for each $\alpha \in \mathbb{C}$ and we leave the proof of the last assertion as an exercise to the reader.

We have already seen that the function $\exp(z^2/2)$ is not in the space $\mathcal{H}(\infty)$ and that is why the function $\exp(z/2)$ is not in the space $\mathcal{L}^{(\alpha)}(\infty)$ for any $\alpha \in \mathbb{R}$. Since $\lim_{\lambda\to 1/2-0}\exp(z/2)$ uniformly on every compact subset of $\mathbb{C}$, the space $\mathcal{L}^{(\alpha)}(\infty)$ considered with the topology induced by that of the space $\mathcal{H}(\mathbb{C})$ is not closed in $\mathcal{H}(\mathbb{C})$, i.e. it is not a Fréchet space.

Remark. Setting $-\zeta/(1-\zeta) = w$ in (3.16), we obtain the representation [II, (2.4)].

3.4 The representation by series in Laguerre polynomials of some classes of holomorphic functions can be obtained without using **(V.3.7)**. Of course, this proposition ensures the existence of the corresponding expansions but the way we are going to follow leads to integral representations of their coefficients in terms of the Laguerre associated functions.

(V.3.9) *Suppose that* $0 < \lambda < \infty, \alpha \in \mathbb{R}\setminus\mathbb{Z}^-$ *and let* F *be a locally L-integrable complex-valued function on the ray* $(-\infty, -\lambda_0^2]$. *If* F *satisfies the condition*:

$$\int_{-\infty}^{-\lambda_0^2} |F(t)|(-t)^{\sigma(\alpha)}\, dt < \infty,\ \sigma(\alpha) = \max(-1, \alpha/2 - 5/4), \tag{3.17}$$

then the function

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{-\lambda_0^2} \frac{F(t)}{t-z}\, dt,\ z \in \mathbb{C}\setminus(-\infty, -\lambda_0^2] \tag{3.18}$$

is representable for $z \in \Delta(\lambda_0)$ *by a series in the Laguerre polynomials* $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ *with coefficients*

$$a_n = \frac{1}{2\pi i I_n^{(\alpha)}}\int_{-\infty}^{-\lambda_0^2} F(t)M_n^{(\alpha)}(t)\, dt,\ n = 0, 1, 2, \dots. \tag{3.19}$$

Proof. From (3.17) and the asymptotic formula [Chapter III, (3.1)] it follows immediately that the integral in the right-hand side of (3.19) is absolutely convergent for $n = 0, 1, 2, \dots$. We replace ζ by t in [Chapter III, (4.10)], multiply by $(2\pi i)^{-1}F(t)$, and integrate on the interval $(-\infty, -\lambda_0^2]$. Thus, we obtain that

$$f(z) = \sum_{n=0}^{\nu} a_n L_n^{(\alpha)}(z) + R_\nu^{(\alpha)}(z),\ z \in \mathbb{C}\setminus(-\infty, -\lambda_0^2],\ \nu = 0, 1, 2, \dots,$$

where $a_n,\ n = 0, 1, 2, \dots$ are given by the equalities (3.19),

$$R_\nu^{(\alpha)}(z) = \frac{1}{2\pi i}\int_{-\infty}^{-\lambda_0^2} \frac{\Delta_\nu^{(\alpha)}(z,t)F(t)}{t-z}\, dt,\ \nu = 0, 1, 2, \dots,$$

and $\Delta^{(\alpha)}(z,t)$ is obtained from [Chapter I, (4.28)] by substituting t for ζ.

The asymptotic formula [Chapter III, (2.1)], the inequality [Chapter III, (5.1)] and the condition (3.17) yield

$$|R_\nu^{(\alpha)}(z)| = O\left(\nu^{1/2}\exp\{-2[\lambda_0 - \mathrm{Re}(-z)^{1/2}]\sqrt{\nu}\}\int_{-\infty}^{-\lambda_0^2} |F(t)|^{\sigma(\alpha)}\, dt\right),$$

for $z \in \Delta(\lambda_0)\setminus[0,\infty)$ and, hence, $\lim_{\nu\to\infty} R_\nu^{(\alpha)}(z) = 0$. This means that the series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ with coefficients (3.19) is convergent in the region $\Delta(\lambda_0)\setminus[0,\infty)$. From **(IV.2.1), (a)** it immediately follows that this series converges in the whole region $\Delta(\lambda_0)$ and, moreover, that it represents the function f there.

If $0 < \lambda_0 < \infty$, then we denote by $V(\lambda_0)$ the half-plane $\operatorname{Re} z > -\lambda_0^2$ and define $V(\infty) = \mathbb{C}$.

(V.3.10) *Suppose that* $0 < \lambda_0 \le \infty, \alpha \in \mathbb{R} \setminus \mathbb{Z}^-$ *and let* $f \in \mathcal{H}(V(\lambda_0))$. *If* $\lim_{z \in V(\lambda), z \to \infty} f(z) = 0$ *for any* $\lambda \in (0, \lambda_0)$, *then the function* f *is representable for* $z \in \Delta(\lambda_0)$ *by a series in the Laguerre polynomials* $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ *with coefficients*

$$a_n = -\frac{1}{2\pi i I_n^{(\alpha)}} \int_{-\lambda^2 - i\infty}^{-\lambda^2 + i\infty} f(\zeta) M_n^{(\alpha)}(\zeta)\, d\zeta,\ \lambda \in (0, \lambda_0),\ n = 0, 1, 2, \dots. \tag{3.20}$$

Proof. The function f has the representation

$$f(z) = -\frac{1}{2\pi i} \int_{-\lambda^2 - i\infty}^{\lambda^2 + i\infty} \frac{f(\zeta)}{\zeta - z}\, d\zeta,\ z \in V(\lambda). \tag{3.21}$$

in any half-plane $V(\lambda), 0 < \lambda < \lambda_0$.

In order to prove this, we denote by $-\lambda^2 \pm \mu(\lambda, R), \mu(\lambda, R) > 0$ the points of intersection of the line $\operatorname{Re} z = -\lambda^2$ and the circle $C(0; R)$ with $R > \max(\lambda^2, |z|)$. If $\Gamma(\lambda, R)$ is the positively oriented arc of the circle $C(0; R)$ lying in the half-plane $V(\lambda)$, then the Cauchy integral formula yields

$$f(z) = -\frac{1}{2\pi i} \int_{-\lambda^2 - i\mu(\lambda, R)}^{-\lambda^2 + i\mu(\lambda, R)} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_{\Gamma(\lambda, R)} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{3.22}$$

If $M(\lambda, R) = \max\{|f(\zeta)| : \zeta \in \Gamma(\lambda, R)\}$, then

$$\left| \frac{1}{2\pi i} \int_{\Gamma(\lambda, R)} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right| = O\left(\frac{R M(\lambda, R)}{R - |z|} \right)$$

and, since $\lim_{R \to \infty} M(\lambda, R) = 0$, the representation (3.21) is a corollary of (3.22).

Further, the Christoffel-Darboux formula [Chapter III, (4.30)] gives

$$\begin{aligned} -\frac{f(\zeta)}{2\pi i(\zeta - z)} &+ \frac{1}{2\pi i I_0^{(\alpha)}} L_0^{(\alpha)}(z) M_0^{(\alpha)}(\zeta) f(\zeta) \\ &= -\sum_0^{\nu} \frac{1}{2\pi i I_n^{(\alpha)}} L_n^{(\alpha)}(z) M_n^{(\alpha)}(\zeta) - \frac{\Delta_\nu(z, \zeta) f(\zeta)}{2\pi i(\zeta - z)}. \end{aligned} \tag{3.23}$$

Using the asymptotic formula [Chapter III, (3.2)] we prove easily that the integral in (3.20) is absolutely convergent for each $n \ge 1$, and then (3.23) yields that it exists for $n = 0$ too. Moreover, by integrating the equality (3.23) on the line $\zeta = -\lambda^2 + i\tau, -\infty < \tau < \infty$, we obtain that for $z \in V(\lambda)$,

$$f(z) = \sum_{n=0}^{\nu} a_n L_n^{(\alpha)}(z) + R_\nu(\alpha; z), \tag{3.24}$$

where $a_n,\ n = 0, 1, 2, \ldots$, are given by the equalities (3.20) and

$$\begin{aligned} R_\nu(\alpha; z) = -\frac{1}{2\pi i} \int_{-\lambda^2 - i\infty}^{-\lambda^2 + i\infty} \frac{\Delta_\nu(z, \zeta) f(\zeta)}{\zeta - z} \, d\zeta \\ = \frac{(\nu+1) L_\nu^{(\alpha)}(z)}{2\pi I_\nu^{(\alpha)}} \int_{-\infty}^{\infty} \frac{M_{\nu+1}^{(\alpha)}(-\lambda^2 + i\tau) f(-\lambda^2 + i\tau)}{z + \lambda^2 - i\tau} \, d\tau \\ - \frac{(\nu+1) L_{\nu+1}^{(\alpha)}(z)}{2\pi I_\nu^{(\alpha)}} \int_{-\infty}^{\infty} \frac{M_\nu^{(\alpha)}(-\lambda^2 + i\tau) f(-\lambda^2 + i\tau)}{z + \lambda^2 - i\tau} \, d\tau \\ = R_\nu^{(1)}(\alpha; z) + R_\nu^{(2)}(\alpha; z). \end{aligned}$$

If $z \in \Delta(\lambda)$, then [Chapter III,(2.3)], [Chapter III, (4.1)] and Stirling's formula yield

$$|R_\nu^{(1)}(\alpha; z)| = O\left\{ \nu^{s(\alpha)} \exp[2 \operatorname{Re}(-z)^{1/2} \sqrt{\nu}] \int_{-\infty}^{\infty} |M_{\nu+1}^{(\alpha)}(-\lambda^2 + i\tau) \, d\tau \right\}, \tag{3.25}$$

where $s(\alpha)$ depends on α only.

As a corollary of the integral representation [Chapter III, (2.4)] we obtain the inequality

$$\begin{aligned} \int_{-\infty}^{\infty} |M_{\nu+1}^{(\alpha)}(-\lambda^2 + i\tau)| \, d\tau \le \int_{-\infty}^{\infty} \left\{ \int_0^{\infty} \frac{t^{\nu+1+\alpha} \exp(-t)}{[(t+\lambda^2)^2 + \tau^2]^{(\nu+2)/2}} \, dt \right\} d\tau \\ = \int_0^{\infty} \left\{ \int_{-\infty}^{\infty} \frac{d\tau}{[(t+\lambda^2)^2 + \tau^2]^{(\nu+2)/2}} \right\} t^{\nu+1+\alpha} \exp(-t) \, dt. \end{aligned}$$

Setting $\tau = (t + \lambda^2)u, -\infty < u < \infty$, we get that

$$\begin{aligned} \int_{-\infty}^{\infty} |M_{\nu+1}^{(\alpha)}(-\lambda^2 + i\tau)| \, d\tau \le \int_0^{\infty} \frac{t^{\nu+1+\alpha} \exp(-t) \, dt}{(t+\lambda^2)^{\nu+1}} \int_{-\infty}^{\infty} \frac{du}{(1+u^2)^{(\nu+2)/2}} \\ \le \int_0^{\infty} \frac{t^{\nu+1+\alpha} \exp(-t) \, dt}{(t+\lambda^2)^{\nu+1}} \int_{-\infty}^{\infty} \frac{du}{1+u^2} = -\pi M_\nu^{\alpha+1}(-\lambda^2). \end{aligned}$$

Further, from the asymptotic formula [Chapter III, (3.1)] as well as from (3.25), we have that $\lim_{\nu\to\infty} R_\nu^{(1)}(\alpha; z) = 0$. In a similar way we prove that

$\lim_{\nu\to\infty} R_\nu^{(2)}(\alpha; z) = 0$ for $z \in \Delta(\lambda)$. Then (3.24) yields that the function f is representable in the region $\Delta(\lambda)$ as a series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ with coefficients given by the equalities (3.20). Since $\lambda \in (0, \lambda_0)$ was arbitrary, the assertion follows from **(IV.5.4)**.

A proposition like **(V.3.10)** holds under weaker hypothesis on the growth of the function f in the half-planes $V(\lambda), 0 < \lambda < \lambda_0$.

(V.3.11) *Suppose that $0 < \lambda_0 \le \infty, \alpha \in \mathbb{R} \setminus \mathbb{Z}^-$ and that $f \in \mathcal{H}(V(\lambda_0))$. If there exists $q \in \mathbb{R}$ such that for each $\lambda \in (0, \lambda_0), |f(z)| = O(|z|^q)$ when $z \to \infty$ in $V(\lambda)$, then the function f is representable as a series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ in the region $\Delta(\lambda_0)$. Moreover, if $k > q+1$ is a positive integer, then the coefficients $\{a_n\}_{n=k}^{\infty}$ of this representation are given by the equalities* (3.20).

Proof. It is easy to prove that if $\lambda \in (0, \lambda_0)$ then

$$f^{(k)}(z) = \frac{k!}{2\pi i}\int_{-\lambda^2-i\infty}^{-\lambda^2+i\infty} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta,\ \ z \in V(\lambda).$$

Differentiating the Christoffel-Darboux formula [Chapter III, (4.12)] with respect to z and using [Chapter IV, (5.8)], we obtain that for $\nu \ge k$

$$\frac{k!}{(\zeta - z)^{k+1}} = \sum_{n=k}^{\nu} \frac{(-1)^k M_n^{(\alpha)}(\zeta)}{I_n^{(\alpha)}} L_{n-k}^{(\alpha+k)}(z) \tag{3.26}$$

$$+\sum_{s=0}^{k} \binom{k}{s} \frac{(-1)^s (k-s)!(\nu+1)}{(\zeta - z)^{k-s+1} I_\nu^{(\alpha)}} \{L_{\nu-s}^{(\alpha+s)}(z) M_{\nu+1}^{(\alpha)}(\zeta) - L_{\nu-s+1}^{(\alpha+s)}(z) M_\nu^{(\alpha)}(\zeta)\}.$$

As in the proof of **(V.3.10)** we obtain the equality

$$f^{(k)}(z) = \sum_{n=k}^{\infty} (-1)^k a_n L_{n-k}^{(\alpha+k)}(z),\ \ z \in \Delta(\lambda),$$

where $\{a_n\}_{n=k}^{\infty}$ are given by (3.20). We integrate k-times the last representation and obtain that $f(z) = P_{k-1}(z) + \displaystyle\sum_{n=k}^{\infty} a_n L_n^{(\alpha)}(z)$, where P_{k-1} is a polynomial of degree less or equal to $k-1$. Notice that the termwise integration is possible since each of the series $\displaystyle\sum_{n=p}^{\infty} a_n L_{n-p}^{(\alpha+p)}(0), p = 0, 1, 2, \dots, k$, is convergent. Indeed, if $\lambda \in (0, \lambda_0)$, then $-\limsup_{n\to\infty} (2\sqrt{n})^{-1} \log |a_n| \ge \lambda$ and, moreover, $L_{n-p}^{(\alpha+p)}(0) = \dbinom{n+\alpha}{n-p} = O(n^{\alpha+p})$ when $n \to \infty$.

Since $\deg L_n^{(\alpha)}(z) = n$ for $\alpha \in \mathbb{C}$ and $n = 0, 1, 2, \dots$, the system of the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ is linearly independent, i.e. it is a basis in the space of all polynomials. In particular, $P_{k-1}(z)$ is a linear combination of the polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{k-1}$.

An example illustrating the proposition **(V.3.12)** is the function $z^p\exp(-\sigma z)$, where p is a positive integer and $\sigma > 0$. It shows that there exists an entire function satisfying the condition $|f(z)| = O(|z|^p)$ in every half-plane $V(\lambda), 0 < \lambda < \infty$, but not in $V(\infty) = \mathbb{C}$.

(V.3.12) *Suppose that* $0 < \lambda < \infty, \alpha \in \mathbb{R}\setminus\mathbb{Z}^-$ *and let* F *be a locally* L-*integrable complex function on the parabola* $p(\lambda)$. *If* F *satisfies the condition*:

$$\int_{p(\lambda)} |\zeta|^{\sigma(\alpha)}|F(\zeta)|dm(\lambda) < \infty,\ \sigma(\alpha) = \max(-1, \alpha/2 - 5/4), \tag{3.27}$$

where $m(\lambda)$ *is the Lebesgue measure on* $p(\lambda)$, *then the function*

$$f(z) = \frac{1}{2\pi i}\int_{p(\lambda)} \frac{F(\zeta)}{\zeta - z}\,d\zeta,\ z \in \mathbb{C}\setminus p(\lambda), \tag{3.28}$$

is representable for $z \in \Delta(\lambda)$ *by a series in the Laguerre polynomials* $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ *with coefficients*

$$a_n = \frac{1}{2\pi i I_n^{(\alpha)}}\int_{p(\lambda)} M_n^{(\alpha)}(\zeta)F(\zeta)\,d\zeta,\ n = 0,1,2,\dots. \tag{3.29}$$

Proof. Notice that the integrals in (3.28) and (3.29) do exist and this follows from (3.27) as well as from the asymptotic formula [Chapter III, (3.2)]. Then the Christoffel-Darboux formula [Chapter III, (4.30)] yields the representation $f(z) = \sum_{n=0}^{\nu} a_n L_n^{(\alpha)}(z) + r_\nu^{(\alpha)}(z)$, where $a_n,\ n = 0,1,2,\dots$, are given by (3.29) and

$$r_\nu^{(\alpha)}(z) = \frac{1}{2\pi i}\int_{p(\lambda)} \frac{\Delta_\nu^{(\alpha)}(z,\zeta)F(\zeta)}{\zeta - z}\,d\zeta, \quad \nu = 0,1,2,\dots.$$

Furthermore, Stirling's formula and the asymptotic formulas for the Laguerre polynomials as well as the inequality [Chapter III, (5.1)] imply that if $z \in \Delta(\lambda)$, then

$$|r_\nu^{(\alpha)}(z)| = O\left\{\nu^{1/2}\exp[-2(\lambda - \mathrm{Re}(-z)^{1/2})\sqrt{\nu}]\int_{p(\lambda)} |\zeta|^{\alpha/2-5/4}|F(\zeta)|dm(\lambda)\right\}$$

and, hence, $\lim_{\nu\to\infty} r_\nu^{(\alpha)}(z) = 0$.

Remark. Schwarz's inequality enables us to replace the hypothesis (3.27) by the following one:

$$\int_{p(\lambda)} |\zeta|^{\max(0,\alpha-1/2)}|F(\zeta)|^2dm(\lambda) < \infty. \tag{3.30}$$

(V.3.13) *Suppose that* $0 < \lambda_0 \le \infty, \alpha \in \mathbb{R} \setminus \mathbb{Z}^-$ *and let the function* $f \in \mathcal{H}(\Delta(\lambda_0))$ *satisfy the conditions:*

(a) *for each* $\lambda \in (0, \lambda_0)$ *there exists* $\delta(\lambda) > 0$ *such that* $|f(z)| = O(|z|^{1/2-\delta(\lambda)})$ *when* $z \to \infty$ *in* $\overline{\Delta}(\lambda)$;

(b) $\displaystyle\int_{p(\lambda)} |z|^{\sigma(\alpha)}|f(z)|dm(\lambda) < \infty,\ 0 < \lambda < \lambda_0,\ \sigma(\alpha) = \max(-1, \alpha/2 - 5/4).$

Then f *is representable for* $z \in \Delta(\lambda_0)$ *by a series in the Laguerre polynomials* $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ *with coefficients*

$$a_n = \frac{1}{2\pi i I_n^{(\alpha)}} \int_{p(\lambda)} M_n^{(\alpha)}(\zeta) f(\zeta)\, d\zeta,\ n = 0, 1, 2, \dots. \tag{3.31}$$

Proof. For $\lambda \in (0, \lambda_0)$ we have the integral representation

$$f(z) = \frac{1}{2\pi i} \int_{p(\lambda)} \frac{f(\zeta)}{\zeta - z}\, d\zeta,\ z \in \Delta(\lambda). \tag{3.32}$$

Indeed, if $z \in \Delta(\lambda)$ is fixed, then for $\rho > \max(1, 2\lambda^2, |z|)$,

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma(\lambda,\rho)} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_{\gamma(\lambda,\rho)} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

where $\Gamma(\lambda, \rho)$ and $\gamma(\lambda, \rho)$ are as in the proof of **(IV.2.1)**. If $l(\lambda, \rho)$ is the length of $\gamma(\lambda, \rho)$, then $l(\lambda, \rho) = O(\sqrt{\rho})$ when $\rho \to \infty$ and, hence,

$$\left| \int_{\gamma(\lambda,\rho)} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right| = O(\rho^{-\delta(\lambda)}) = o(1),\ \rho \to \infty.$$

From **(V.3.12)** it follows that for $\lambda \in (0, \lambda_0)$ the function f is representable in $\Delta(\lambda)$ by a series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ with coefficients given by the equalities (3.31). Therefore, the assertion we wish to prove is a corollary of **(IV.5.4)**.

Remarks: (1) It is clear that the condition **(a)** can be replaced by the requirement

$$\int_{\gamma(\lambda,\rho)} |f(\zeta)| dm(\lambda) = o(\rho),\ \rho \to \infty. \tag{3.33}$$

(2) It seems that the conditions **(a)** (respectively (3.33)) and **(b)** are independent. For instance, if $\alpha = 1/2$, then the function $f(z) = (z + \lambda_0^2)^{\delta}, 0 < \lambda_0 < \infty$, with $0 < \delta < 1/2$ satisfies **(a)** but not **(b)**. It is not clear, in general, whether the validity of **(b)** implies **(a)** (or (3.33), respectively).

(V.3.14) *Suppose that $0 < \lambda_0 \le \infty, \alpha > -1$ and $f \in \mathcal{H}(\Delta(\lambda_0))$. If for each $\lambda \in (0, \lambda_0)$ there exists $\delta(\lambda) > 0$ such that $|f(z)| = O(|z|^{-1/2-\delta(\lambda)})$ when $z \to \infty$ in $\overline{\Delta}(\lambda)$, then the function f is representable for $z \in \Delta(\lambda_0)$ by a series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ with coefficients given by* (3.29).

Proof. Since the function f is in the space $\mathcal{L}(\lambda_0)$, from **(V.3.6)** and **(V.3.7)** it follows that it has an expansion for $z \in \Delta(\lambda_0)$ in a series of Laguerre polynomials with parameter α. Moreover, **(IV.5.3)** and [Chapter I, (3.1)] yield that the coefficients of this expansion are

$$a_n = \frac{1}{n! I_n^{(\alpha)}} \int_0^{\infty} \{t^{n+\alpha} \exp(-t)\}^{(n)} f(t)\, dt,\ n = 0, 1, 2, \ldots .$$

It is easy to see that if $\zeta = \xi + i\eta \in p(\lambda)$ and $0 \le t < \infty$ then $|\zeta - t|^2 \ge \xi^2 + \eta^2 = (\xi + 2\lambda^2)^2$ if $\xi \le 0$ and $|\zeta - t|^2 \ge \eta^2 = 4\lambda^2(\xi + \lambda^2)$ for $\xi \ge 0$. Therefore,

$$\int_{p(\lambda)} \frac{ds}{|\zeta - t|^{2n+2}}$$

$$\le 2 \int_{-\lambda^2}^{0} \frac{((\xi + 2\lambda^2)/(\xi + \lambda^2))^{1/2}}{(\xi + 2\lambda^2)^{n+1}}\, d\xi + 2 \int_0^{\infty} \frac{((\xi + 2\lambda^2)/(\xi + \lambda^2))^{1/2}}{(4\lambda^2(\xi + \lambda^2))^{n+1}}\, d\xi$$

for $n = 0, 1, 2, \ldots$.

For $\lambda \in (0, \lambda_0)$ we have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{p(\lambda)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta,\ n = 0, 1, 2, \ldots \tag{3.34}$$

provided $z \in \Delta(\lambda)$.

In particular, if $z = t \in [0, \infty)$, then Schwarz's inequality gives

$$|f^{(n)}(t)|^2 \le \Big(\frac{n!}{2\pi}\Big)^2 \int_{p(\lambda)} |f(\zeta)|^2\, ds \int_{p(\lambda)} \frac{ds}{|\zeta - t|^{2n+2}} \le K(\lambda, n),$$

where $K(\lambda, n)$ is a constant depending on λ and n only. Then, after integration by parts, we obtain that

$$a_n = \frac{(-1)^n}{n! I_n^{(\alpha)}} \int_0^{\infty} t^{n+\alpha} \exp(-t) f^{(n)}(t)\, dt,\ n = 0, 1, 2, \ldots,$$

and, having in view (3.34), we come to the following integral representation of the coefficients $a_n,\ n = 0, 1, 2, \ldots,$

$$a_n = -\frac{1}{2\pi i I_n^{(\alpha)}} \int_0^{\infty} t^{n+\alpha} \exp(-t) \Big\{ \int_{p(\lambda)} \frac{f(\zeta)}{(t - \zeta)^{n+1}}\, d\zeta \Big\}\, dt. \tag{3.35}$$

Since

$$\int_0^\infty t^{n+\alpha}\exp(-t)\,dt\int_{p(\lambda)}\frac{|f(\zeta)|}{|t-\zeta|^{n+1}}\,ds<\infty,$$

the multiple integral in (3.35) is absolutely convergent and changing the order of integrations we obtain finally ($n=0,1,2,\dots$)

$$a_n=-\frac{1}{2\pi i I_n^{(\alpha)}}\left\{\int_0^\infty\frac{t^{n+\alpha}\exp(-t)}{(t-\zeta)^{n+1}}\,dt\right\}f(\zeta)\,d\zeta=\frac{1}{2\pi i I_n^{(\alpha)}}\int_{p(\lambda)}M_n^{(\alpha)}(\zeta)f(\zeta)\,d\zeta.$$

Remark. If $\alpha\le 3/2$, then the proposition **(V.3.14)** is a particular case of **(V.3.13)** since $\sigma(\alpha)\le -1/2$.

3.5 So far we have considered series expansions in Laguerre polynomials of holomorphic functions defined by means of Caychy type integrals, and we have got integral representations of the coefficients of these expansions in terms of Laguerre's associated functions. Now we shall see that under certain conditions the existence of such representations is equivalent to the representations of the sums of the series into consideration as Cauchy type integrals.

For a function $f\in\mathcal{H}(\Delta(\lambda_0)),0<\lambda_0\le\infty$, we say that it is representable as a Caychy type integral in the region $\Delta(\lambda_0)$, if the equality (3.33) holds for every $\lambda\in(0,\lambda_0)$ and $z\in\Delta(\lambda)$.

(V.3.15) *Suppose that* $0<\lambda_0\le\infty,\alpha\in\mathbb{R}\setminus\mathbb{Z}^-$ *and let* $f\in\mathcal{H}(\Delta(\lambda_0))$ *be such that*

$$\int_{p(\lambda)}|\zeta|^{\alpha/2-1/4}|f(\zeta)|\,ds<\infty \tag{3.37}$$

for $\lambda\in(0,\lambda_0)$. *Then in order that* f *has an expansion for* $z\in\Delta(\lambda_0)$ *in a series of the Laguerre polynomials* $\{L_n^{(\alpha)}(z)\}_{n=0}^\infty$ *with coefficients given by the equalities* (3.32), *it is necessary and sufficient* f *to be representable as a Cauchy type integral in the region* $\Delta(\lambda_0)$.

Proof. From he asymptotic formula [Chapter III,(2.3)], the inequalities [Chapter III, (4.1)], [Chapter III, (5.1)] and Stirling's formula we obtain that for each $\lambda\in(0,\lambda_0)$ and $z\in\Delta(\lambda)$,

$$(I_n^{(\alpha)})^{-1}|L_n^{(\alpha)}(z)M_n^{(\alpha)}(\zeta)|=O\{n^{a(\alpha)}|\zeta|^{\alpha/2-1/4}\exp[-2(\lambda-\mathrm{Re}(-z)^{1/2})\sqrt{n}]\}, \tag{3.38}$$

uniformly with respect to $\zeta\in p(\lambda)$, where $a(\alpha)$ depends on α only. This estimate yields the following representation of the Cauchy kernel:

$$\frac{1}{\zeta-z}=\sum_{n=0}^\infty\frac{1}{I_n^{(\alpha)}}L_n^{(\alpha)}(z)M_n^{(\alpha)}(\zeta),\ \ z\in\Delta(\lambda),\zeta\in p(\lambda). \tag{3.39}$$

Moreover, if z is fixed, then the series in the right-hand side of (3.39) is uniformly convergent on each finite arc of $p(\lambda)$.

Suppose that the function $f \in \mathcal{H}(\Delta(\lambda_0))$ satisfies (3.37) and that it has Cauchy type integral representation in the region $\Delta(\lambda_0)$. Then from (3.38) it follows that after multiplying the series in (3.39) by $f(\zeta)$ we obtain a series which can be integrated term by term on $p(\lambda)$. Therefore, f has an expansion in a series of the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ in $\Delta(\lambda)$, with coefficients given by the equalities (3.32). Since this is true for any $\lambda \in (0, \lambda_0)$, the sufficiency is proved.

Conversely, suppose that (3.37) holds for the function $f \in \mathcal{H}(\Delta(\lambda_0))$ and that it is representable for $z \in \Delta(\lambda_0)$ by a series in the polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ with coefficients (3.32). Then by means of (3.38) we come to the conclusion that for the equality (3.33) holds for $\lambda \in (0, \lambda_0)$ and $z \in \Delta(\lambda)$, i.e. f is representable as a Cauchy type integral in the region $\Delta(\lambda_0)$.

4. Representations by series in Laguerre and Hermite associated functions

4.1 Suppose that $0 \le \mu_0 < \infty, \alpha \in \mathbb{R}\setminus\mathbb{Z}^-$ and denote by $\mathcal{M}^{(\alpha)}(\mu_0)$ the $\mathbb{C}$-vector space of the complex functions which are holomorphic in the region $\Delta^*(\mu_0)$ and are representable there by series in the Laguerre associated functions $\{M_n^{(\alpha)}(z)\}_{n=0}^{\infty}$. Since the last system has the uniqueness property [**(IV.5.10)**], it is a basis in each space $\mathcal{M}^{(\alpha)}(\mu_0)$ with $0 \le \mu_0 < \infty$ and $\alpha \in \mathbb{R}\setminus\mathbb{Z}^-$. It is easy to prove that, in fact, $\mathcal{M}^{(\alpha)}(\mu_0)$ is a proper (vector) subspace of $\mathcal{H}(\Delta^*(\mu_0))$. This assertion is a corollary of the following proposition:

<u>**(V.4.1)**</u> *If* $0 \le \mu_0 < \infty, \alpha \in \mathbb{R}\setminus\mathbb{Z}^-$ *and* $f \in \mathcal{M}^{(\alpha)}(\mu_0)$, *then for* $\mu \in (\mu_0, \infty)$ *and* $z \in \overline{\Delta^*(\mu)}$,

$$|f(z)| = O(|z|^{\max(-1,\alpha/2-1/4)}). \tag{4.1}$$

Proof. Suppose that the representation [Chapter III, (4.17)] holds for the function $f \in \mathcal{H}(\Delta^*(\mu_0))$ in the region $\Delta^*(\mu_0)$. If $\mu_0 < \mu < \infty$ and $0 < \varepsilon < \mu - \mu_0$, then there exists a constant B such that $|b_n| \le B\exp[2(\mu_0+\varepsilon)\sqrt{n}]$, $n = 0, 1, 2, \ldots$. If $k \ge \nu(\alpha) = \max(1, -\alpha/2-3/4, -\alpha-1)$ is an integer, then the asymptotic formula [Chapter III, (4.17] and the inequality [Chapter III, (5.1)] give that for $z \in \overline{\Delta^*(\mu)}$,

$$|f(z)| \le \sum_{n=0}^{k} |b_n||M_n^{(\alpha)}(z)| + \sum_{n=k+1}^{\infty} |b_n||M_n^{(\alpha)}(z)|$$

$$= O(|z|^{-1}) + O(|z|^{\alpha/2-1/4} \sum_{n=k+1}^{\infty} \exp[-2(\mu-\mu_0-\varepsilon)\sqrt{n}])$$

$$= O(|z|^{\max(-1,\alpha/2-1/4)}).$$

Remark. A growth characteristic of the functions in the space $\mathcal{M}^{(\alpha)}(\mu_0)$ is not known at the present. The solution of this problem, i.e. the "discovery" of a proposition like **(V.3.7)** is, undoubtly, of considerable interest.

4.2 The series representations in Laguerre associated functions of some classes of holomorphic functions can be obtained by means of the asymptotic formulas for the Laguerre systems and the Christoffel-Darboux formula for these systems. In particular, for the expansions of functions defined by Cauchy type integrals is typical that their coefficients can be expressed in terms of Laguerre polynomials.

(V.4.2) *Suppose that $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$ and let F be a locally L-integrable complex function on the ray $[0, \infty)$. If $|F(t)| = O(t^\beta \exp(-t/2))$ for some $\beta \le (\alpha - 1)/2$ when $t \to \infty$, then the function*

$$f(z) = -\frac{1}{2\pi i} \int_0^\infty \frac{F(t)}{t - z}\, dt,\ \ z \in \mathbb{C} \setminus [0, \infty) \tag{4.2}$$

is representable for $z \in \mathbb{C} \setminus [0, \infty) = \Delta^*(0)$ as a series in the Laguerre associated functions $\{M_n^{(\alpha)}(z)\}_{n=0}^\infty$ with coefficients

$$b_n = \frac{1}{2\pi i I_n^{(\alpha)}} \int_0^\infty L_n^{(\alpha)}(t) F(t)\, dt,\ \ n = 0, 1, 2, \dots. \tag{4.3}$$

Proof. From [Chapter III, (4.30)] it follows that if $z \in \Delta^*(0)$, then

$$f(z) = \sum_{n=0}^{\nu} b_n M_n^{(\alpha)}(z) + \frac{1}{2\pi i} \int_0^\infty \frac{\Delta_\nu^{(\alpha)}(t, z) F(t)}{z - t}\, dt,\ \ \nu = 0, 1, 2, \dots, \tag{4.4}$$

where $\Delta^{(\alpha)}(t, z)$ is obtained from [Chapter I,(4.28)] by substituting t for z and z for ζ.

By means of [Chapter III, (4.2)] and [Chapter III, (4.1)], respectively, we obtain that if $\omega \ge 1$ and $z \in \Delta^*(0)$, then

$$\int_\omega^\infty \left| \frac{L_n^{(\alpha)}(t) F(t)}{t - z} \right| dt = O\left(n^{\alpha/2 + 1/6} \int_\omega^\infty t^{-2}\, dt \right)$$

and

$$\int_0^\omega \left| \frac{L_n^{(\alpha)}(t) F(t)}{t - z} \right| dt = O\left(n^{a(\alpha)} \int_0^\omega |F(t)|\, dt \right), \tag{4.5}$$

where $a(\alpha) = \max(\alpha, \alpha/2 - 1/4)$. Further, the asymptotic formula [Chapter III, (4.16)] as well as Stirling's formula give that if $z \in \Delta^*(0)$, then

$$\left| \int_0^\infty \frac{\Delta_\nu^{(\alpha)}(t, z) F(t)}{t - z}\, dt \right| = O\left\{ \nu^{s(\alpha)} \exp[-2 \operatorname{Re}(-z)^{1/2} \sqrt{\nu}] \right\},$$

where $s(\alpha)$ depends on α only. The assertion follows from (4.4).

It seems that there is another way to prove a proposition like **(V.4.2)**. More precisely, let $F(t) = t^{\alpha}\exp(-t)\varphi(t), \alpha > -1$ and let

$$\varphi(t) = \sum_{n=0}^{\infty} b_n L_n^{(\alpha)}(t),\ 0 < t < \infty. \tag{4.6}$$

Then we multiply by $t^{\alpha}\exp(-t)(z-t){-1}, z \in \mathbb{C} \setminus [0,\infty)$, integrate on the ray $[0,\infty)$, and having in mind [Chapter III, (4.12)], we arrive to a representation of the function (4.2) by a series in Laguerre associated functions.

Now we shall see that in this way it is not possible to obtain a generalization of **(V.4.2)**. Moreover, there are cases when the approach, just described, is not applicable but the hypotheses of **(V.4.2)** are still satisfied.

Suppose that $\alpha > 1/2$ and $\varphi(t) = t^{\gamma}\exp(t/2), 0 < t < \infty$, where $-1-\alpha < \gamma < -\alpha/2 - 5/4$. For the coefficients of the series (4.6), i.e.

$$\begin{aligned} b_n &= \frac{1}{I_n^{(\alpha)}} \int_0^{\infty} t^{\alpha}\exp(-t) L_n^{(\alpha)}(t)\varphi(t)\,dt \\ &= \frac{1}{I_n^{(\alpha)}} \int_0^{\infty} t^{\alpha+\gamma}\exp(-t/2) L_n^{(\alpha)}(t)\,dt,\ n = 0,1,2,\dots, \end{aligned} \tag{4.7}$$

we have the asymptotic formula [G. Szegö, 1, (9.5.22)]

$$b_n = \{An^{-\alpha-\gamma-1} + B(-1)^n n^{\gamma}\}(1+\beta_n), \tag{4.8}$$

where $A \neq 0$ and $\lim_{n\to\infty}\beta_n = 0$.

Since $-\alpha/2-\gamma-5/4 > 0$ and $\alpha/2+\gamma-1/4 < 0$, from (4.8) and the asymptotic formula [Chapter III,(4.13)] it follows that the series (4.6) diverges for every $t > 0$. But the function $F(t) = t^{\alpha}\exp(-t)\varphi(t) = t^{\alpha+\gamma}\exp(-t/2)$ satisfies the condition of proposition **(V.4.2)** since $\alpha+\gamma < \alpha/2 - 5/4 < (\alpha-1)/2$.

(V.4.3) *Suppose that $0 < \mu_0 < \infty, \alpha \in \mathbb{R} \setminus \mathbb{Z}^-$ and let F be a complex-valued function satisfying the conditions of proposition* **(V.4.2)** *on the ray $[-\mu_0^2,\infty)$. Then the function*

$$f(z) = -\frac{1}{2\pi i}\int_{-\mu_0^2}^{\infty} \frac{F(t)}{t-z}\,dt,\ z \in \mathbb{C} \setminus [-\mu_0^2,\infty) \tag{4.9}$$

is representable for $z \in \Delta^*(\mu_0)$ by a series in the Laguerre associated functions $\{M_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ with coefficients

$$b_n = \frac{1}{2\pi i I_n^{(\alpha)}}\int_{-\mu_0^2}^{\infty} L_n^{(\alpha)}(t)F(t)\,dt,\ n = 0,1,2,\dots. \tag{4.10}$$

Proof. Using the Christoffel-Darboux formula for the Laguerre systems, we obtain that for $z \in \mathbb{C} \setminus [-\mu_0^2, \infty)$,

$$f(z) = \sum_{n=0}^{\nu} b_n M_n^{(\alpha)}(z) + \frac{1}{2\pi i} \int_{-\mu_0^2}^{\infty} \frac{\Delta_\nu^{(\alpha)}(t,z)F(t)}{z-t}\, dt, \ \nu = 0,1,2,\dots. \tag{4.11}$$

If $\omega > \max(1, 2\mu_0^2)$, then from the asymptotic formula [Chapter III, (2.3)] and the inequality [Chapter III, (4.21)] it follows that

$$\max_{z \in \partial\Delta(\mu_0,\omega)} |L_n^{(\alpha)}(z)| = O(n^{\alpha/2-1/4} \exp(2\mu_0\sqrt{n})),$$

where $\Delta(\mu_0, \omega) = \{z \in \mathbb{C} : \mathrm{Re}(-z) < \mu_0, |z| < \omega\}$.

Further, the maximum modulus principle yields that the above estimate holds for $z = t \in (-\mu_0^2, \omega)$ and then, having in mind (4.5), we obtain

$$\int_{-\mu_0^2}^{\omega} \left| \frac{L_n^{(\alpha)}(t)F(t)}{z-t} \right| dt = O\left(n^{a(\alpha)} \exp(2\mu_0\sqrt{n}) \int_{-\mu_0^2}^{\omega} |F(t)|\, dt \right).$$

As in the proof of **(V.4.2)** we obtain that if $z \in \Delta^*(\mu_0)$, then

$$\int_{-\mu_0^2}^{\infty} \left| \frac{\Delta_\nu^{(\alpha)}(t,z)F(t)}{z-t} \right| dt = O(\nu^{s(\alpha)} \exp[-2(\mathrm{Re}(-z)^{1/2} - \mu_0)\sqrt{\nu}])$$

and the assertion follows from (4.11).

(V.4.4) *Suppose that* $0 < \mu_0 < \infty, \alpha \in \mathbb{R} \setminus \mathbb{Z}^-$ *and let* F *be a locally* L*-integrable complex function on* $p(\mu_0)$ *such that* $F(\zeta) = O((-\zeta)^\beta \exp(-\zeta))$ *for some* $\beta < \alpha/2 + 1/4$ *when* ζ *tends to infinity. Then the function*

$$f(z) = -\frac{1}{2\pi i} \int_{p(\mu_0)} \frac{F(\zeta)}{\zeta - z}\, d\zeta, \ z \in \mathbb{C} \setminus p(\mu_0)$$

is representable for $z \in \Delta^*(\mu_0)$ *by a series in the Laguerre associated functions* $\{M_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ *with coefficients*

$$b_n = \frac{1}{2\pi i I_n^{(\alpha)}} \int_{p(\mu_0)} L_n^{(\alpha)}(\zeta)F(\zeta)\, d\zeta, \ n = 0,1,2,\dots. \tag{4.12}$$

Proof. We give only an outline. If $\rho > \max(1, 2\mu_0^2)$ and $\Gamma(\mu_0, \rho)$ is the arc of $p(\mu_0)$ lying in the closed disk $\{z \in \mathbb{C} : |z| \le \rho\}$, then the inequality [IV,(4.3)] yields that

$$|L_n^{(\alpha)}(\zeta)F(\zeta)| = O(n^{\alpha/2-1/4}|\zeta|^{\beta-\alpha/2-1/4} \exp(2\mu_0\sqrt{n}))$$

uniformly with respect to $\zeta \in p(\mu_0) \setminus \Gamma(\mu_0, \rho)$ when n tends to infinity. Therefore, if $z \notin p(\mu_0)$, then

$$\int_{p(\mu_0)} \left| \frac{L_n^{(\alpha)}(\zeta)F(\zeta)}{\zeta - z} \right| dm(\mu_0) = O\Big(n^{\alpha/2-1/4} \exp(2\mu_0\sqrt{n}) \int_{p(\mu_0)} |\zeta|^{\beta-\alpha/2-3/4}\, dm(\mu_0).$$

Further, the asymptotic formula for the Laguerre polynomials in the region $\mathbb{C} \setminus [0, \infty)$ yields

$$\int_{\Gamma(\mu_0,\rho)} \left| \frac{L_n^{(\alpha)} F(\zeta)}{\zeta - z} \right| dm(\mu_0) = O\Big(n^{\alpha/2-1/4} \exp(2\mu_0\sqrt{n})) \int_{\Gamma(\mu_0,\rho)} |F(\zeta)|\, dm(\mu_0)\Big)$$

provided $z \notin p(\mu_0)$.

The assertion follows from the Christoffel-Darboux formula for the Laguerre systems. Indeed, if $z \in \Delta^*(\mu_0)$, then

$$\int_{p(\mu_0)} \left| \Delta_\nu^{(\alpha)}(\zeta, z)F(\zeta)\zeta - z \right| dm(\mu_0) = O(\nu^{1/2} \exp\{-2(\mathrm{Re}(-z))^{1/2} - \mu_0\sqrt{\nu}\}).$$

4.3 Recall that if $\tau \in \mathbb{R}$, then by $H^+(\tau)$ $(H^-(\tau))$ we denoted the half-plane $\mathrm{Im}\, z > \tau$ $(\mathrm{Im}\, z < \tau)$. If $0 \le \tau_0 < \infty$, then by $\mathcal{G}^+(\tau_0)$ $(\mathcal{G}^-(\tau_0))$ we denote the $\mathbb{C}$-vector space of complex functions which are holomorphic in the half-plane $H^+(\tau_0)$ $(H^-(-\tau_0))$ and have there expansions in series of the Hermite associated functions $\{G_n^+(z)\}_{n=0}^\infty$ $(\{G_n^-(z)\}_{n=0}^\infty)$. Since each of the last systems has the uniqueness property, $\{G_n^+(z)\}_{n=0}^\infty$ is a basis in $\mathcal{G}^+(\tau_0)$ and $\{G_n^-(z)\}_{n=0}^\infty$ is a basis in $\mathcal{G}^-(\tau_0)$.

By the aid of the inequality [Chapter III, (5.4)] it is easy to prove that $\mathcal{G}^+(\tau_0)$ $(\mathcal{G}^-(\tau_0))$ is a proper $\mathbb{C}$-vector subspace of $\mathcal{H}(H^+(\tau_0))$ $(\mathcal{H}(H^-(\tau_0)))$. More precisely, we have the following proposition:

(V.4.5) *The functions $f \in \mathcal{G}^\pm(\tau_0), 0 \le \tau_0 < \infty$, are bounded in the half-plane $H^\pm(\pm\tau)$, whenever $\tau \in (\tau_0, \infty)$.*

A growth characteristic of the functions in the spaces $\mathcal{G}^\pm(\tau_0), 0 \le \tau_0 < \infty$, is not known at present. The Christoffel-Darboux formula for the Hermite systems as well as the asymptotic formulas for these systems give the possibility to obtain the representations by series in Hermite associated functions of some classes of complex functions which are holomorphic in half-planes of the kind $H^\pm(\tau_0), 0 \le \tau_0 < \infty$. A typical example of this kind is the following proposition:

(V.4.6) *Suppose that F is a locally L-integrable function on the real axis and, moreover, $|F(t)| = O(|t|^{-\delta} \exp(-t^2/2))$ for some $\delta > 1$ when $|t|$ tends to infinity. Then the function*

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(t)}{t - z}\, dt,\ z \in \mathbb{C} \setminus \mathbb{R} \tag{4.13}$$

is representable for $z \in H^{\pm}(0)$ *by a series in the Hermite associated functions* $\{G_n^{\pm}(z)\}_{n=0}^{\infty}$ *with coefficients*

$$b_n = \frac{1}{2\pi i I_n} \int_{-\infty}^{\infty} H_n(t)F(t)\,dt,\ n = 0, 1, 2, \ldots. \tag{4.14}$$

Proof. If $T \in (0, \infty)$, then the asymptotic formula [III, (2.6)] yields

$$\int_{|t|\le T} |H_n(t)F(t)|\,dt = O\biggl((2n/e)^{n/2} \int_{|t|\le T} |F(t)|\,dt\biggr).$$

From the inequality [Chapter III, (4.5)] and the Stirling formula it follows that

$$\int_{|t|\ge T} |H_n(t)F(t)|\,dt = O\biggl(n^{1/4}(2n/e)^{n/2} \int_{|t|\ge T} |t|^{-\delta}\,dt\biggr).$$

Therefore,

$$\int_{-\infty}^{\infty} |H_n(t)F(t)|\,dt = O(n^{1/4}(2n/e)^{n/2})$$

when n tends to infinity, and the asymptotic formulas [Chapter III, (3.2)] and [Chapter III, (3.6)] give

$$\int_{-\infty}^{\infty} \frac{\Delta_\nu(t,z)F(t)}{z-t}\,dt = \frac{1}{2I_\nu}\int_{-\infty}^{\infty} \{H_\nu(t)G_{\nu+1}^{\pm}(z) - H_{\nu+1}(t)G_\nu^{\pm}(z)\}\frac{F(t)}{z-t}\,dt$$

$$= O(\nu^{1/4}\exp(-|\operatorname{Im} z|\sqrt{2\nu+1})),\ \nu \to \infty.$$

The assertion is a corollary of the Christoffel-Darboux formula for the Hermite system.

5. Holomorphic extension

5.1 We say that a complex function φ defined on an interval (a, b), $-\infty \le a < b \le \infty$, has a holomorphic extension in the complex plane if there exist a domain $D \subset \mathbb{C}$ containing the interval (a, b), and a complex function $\Phi \in \mathcal{H}(D)$ such that $\Phi(x) = \varphi(x)$ a.e. (almost everywhere) in (a, b). Evidently, the uniqueness of a holomorphic extension follows immediately from the identity theorem for holomorphic functions.

Examples. **(1)** Suppose that $\varphi \in \mathcal{C}^\infty((a, b))$ and that for every compact interval $I \subset (a, b)$ there exist positive $K = K(I)$ and $r = r(I)$ such that $|\varphi^{(n)}(x)| \le Kn!r^{-n}$ for $x \in I$ and $n = 0, 1, 2, \ldots$. Then, as it is well-known, φ has a holomorphic extension in the complex plane.

(2) Suppose that $\varphi \in \mathcal{C}((a, b))$ and let $E_n(\varphi; I)$ be the best approximation of φ on the interval $I \subset (a, b)$ by means of (algebraic) polynomials of degree not greater than n. If for every such I there exist positive $Q = Q(I)$ and $q = q(I) < 1$ such that $E_n(\varphi; I) \le Qq^n$ for $n = 0, 1, 2, \ldots$, then φ has also holomorphic extension.

Now we are going to state sufficient conditions for measurable functions of one real variable to be holomorphically extendable in the complex plane. We consider the cases: I) $a = -1, b = 1$; II) $a = 0, b = \infty$; III) $a = -\infty, b = \infty$ and we refer our requirements to the behaviour of the functions under consideration at the ends of these intervals as well as to the asymptotics of their Fourier-Jacobi, Fourier-Laguerre and Fourier-Hermite coefficients, respectively.

In order to give concise formulations of the corresponding propositions, we introduce the following three classes of functions:

(I) The class $U(\alpha, \beta)(\alpha > -1, \beta > -1)$ consists of all complex-valued measurable functions f on $(-1, 1)$ such that

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}|f(x)|\,dx < \infty.$$

(II) The class $V(\alpha, \delta, r)(\alpha > -1, \delta < 1, r > 0)$ consists of all complex-valued measurable functions g on $(0, \infty)$ such that the function $\exp(-\delta x)g(x)$ is essentially bounded on (r, ∞) and, moreover,

$$\int_{0}^{r} x^{\alpha}|g(x)|\,dx < \infty.$$

(III) The class $W(\delta, r)(\delta < 1, r > 0)$ consists of all complex-valued measurable functions h on $(-\infty, \infty)$ such that the function $\exp(-\delta x^2)h(x)$ is essentially bounded when $|x| \ge r$ and, moreover,

$$\int_{-r}^{r} |h(x)|\,dx < \infty.$$

(V.5.1) *Suppose that* $f \in U(\alpha, \beta)$ *and define*

$$a_n^{(\alpha,\beta)}(f) = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) f(x)\,dx, \ n = 0, 1, 2, \ldots. \tag{5.1}$$

If $R(f) := \{\limsup_{n\to\infty} |a_n^{(\alpha,\beta)}(f)|^{1/n}\}^{-1} < 1$, *then* f *has a holomorphic extension. More precisely, there exists a complex-valued function* F *which is holomorphic in the region* $E(R(f))$ *and such that* $F(x) = f(x)$ *a.e. in* $(-1, 1)$.

Proof. The function $F(z) = \sum_{n=0}^{\infty} a_n^{(\alpha,\beta)}(f) P_n^{(\alpha,\beta)}(z)$ is holomorphic inside the ellipse $E(R(f))$. Moreover, if we define

$$A^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}(F(x) - f(x))$$

for $x \in (-1,1)$, then due to (5.1) we have

$$\int_{-1}^{1} A^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\, dx = 0,\ n = 0,1,2,\dots. \tag{5.2}$$

Since $\alpha+\beta+2>0$, $\deg P_n^{(\alpha,\beta)} = n$ for each $n = 0,1,2,\dots$ and, hence, the system of Jacobi polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ is a basis in the space of all (algebraic) polynomials. Then the equalities (5.2) yield that $\displaystyle\int_{-1}^{1} A^{(\alpha,\beta)}(x) x^n\, dx = 0$ for $n = 0,1,2,\dots$ and, therefore, as it is well-known, $A^{(\alpha,\beta)}(x) = 0$ a.e. in $(-1,1)$, i.e. $F(x) = f(x)$ a.e. in $(-1,1)$.

(V.5.2) *Suppose that $g \in V(\alpha,\delta,r)$ and define*

$$b_n^{(\alpha)}(g) = \int_0^{\infty} x^{\alpha} \exp(-x) L_n^{(\alpha)}(x) g(x)\, dx,\ n = 0,1,2,\dots. \tag{5.3}$$

If $\lambda_0(g) := -\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|b_n^{(\alpha)}| > 0$, then g has a holomorphic extension. More precisely, there exists a complex function G which is holomorphic in the region $\Delta(\lambda_0(g))$ and such that $G(x) = g(x)$ a.e. in $(0,\infty)$. Moreover, for every $\lambda \in [0,\lambda_0)$ there exists a positive constant $M(\lambda) < \infty$ such that the inequality

$$|g(x)| \le M(\lambda)\exp(x/2 - \lambda\sqrt{x}) \tag{5.4}$$

holds a.e. in $(0,\infty)$.

Proof. If we define $B_n^{(\alpha)}(g) = (I_n^{(\alpha)})^{-1} b_n^{(\alpha)}(g),\ n = 0,1,2,\dots$, then Stirling's formula gives that

$$-\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|B_n^{(\alpha)}(g)| = -\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|b_n^{(\alpha)}(g)|.$$

From the Cauchy-Hadamard formula for series in Laguerre polynomials we obtain that the complex function $G(z) = \displaystyle\sum_{n=0}^{\infty} B_n^{(\alpha)}(g) L_n^{(\alpha)}(z)$ is holomorphic in the region $\Delta(\lambda_0(g))$. Moreover,

$$B_n^{(\alpha)}(g) = (I_n^{(\alpha)})^{-1}\int_0^{\infty} x^{\alpha}\exp(-x) L_n^{(\alpha)}(x) G(x)\, dx,\ n = 0,1,2,\dots. \tag{5.5}$$

We define $B^{(\alpha)}(x) = x^{\alpha}\exp(-x)(G(x) - g(x))$ for $x \in (0,\infty)$ and then the equalities (5.3) and (5.4) yield

$$\int_0^{\infty} B^{(\alpha)}(x) L_n^{(\alpha)}(x)\, dx = 0,\ n = 0,1,2,\dots.$$

Since $\deg L_n^{(\alpha)} = n,\ n = 0, 1, 2, \ldots,$ for every $\alpha \in \mathbb{C}$, the system of Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ is a basis in the space of the polynomials and, hence,

$$\int_0^\infty B^{(\alpha)}(x)x^n\,dx = 0,\ n = 0, 1, 2, \ldots. \tag{5.6}$$

Since $\alpha > -1$, **(V.3.6)** implies that the function G is in the space $\mathcal{L}(\lambda_0(g))$, i.e. for every $\lambda \in [0, \lambda_0(g))$ there exists a positive constant $M = M(\lambda)$ such that

$$|G(z)| = |G(x + iy)| \le M(\lambda)\exp\{\varphi(\lambda; x, y)\} \tag{5.7}$$

provided $z \in \overline{\Delta}(\lambda)$. In particular, $|G(x)| \le M(0)\exp(x/2)$ for $x \in [0, \infty)$ and we obtain that $|B^{(\alpha)}(x)| \le B\exp(-(1 - \sigma)x)$ for $x \in (r, \infty)$, where B is a constant and $\sigma = \max(1/2, \delta)$. Therefore, the Fourier transform

$$\hat{B}^{(\alpha)}(w) = \int_0^\infty B^{(\alpha)}(x)\exp(iwx)\,dx$$

of $B^{(\alpha)}(x)$ is a complex-valued function which is holomorphic in the strip $S(1-\sigma)$. Moreover, from (5.6) it follows that the function $\hat{B}^{(\alpha)}(w)$ and all its derivatives vanish at the point $w = 0$. Hence, by the identity theorem for holomorphic functions, $\hat{B}^{(\alpha)}(w) = 0$ for $w \in S(1 - \sigma)$. Then the uniqueness property of the Fourier transform yields that $B^{(\alpha)}(x) = 0$ a.e. in $(0, \infty)$, i.e. $G(x) = g(x)$ a.e. in $(0, \infty)$. It remains to note that (5.4) follows from (5.7).

(V.5.3) *Suppose that $h \in W(\delta, r)$ and define*

$$c_n(h) = \int_{-\infty}^\infty \exp(-x^2)H_n(x)h(x)\,dx,\ n = 0, 1, 2, \ldots.$$

If $\tau_0(h) := -\limsup_{n\to\infty}(2n + 1)^{-1/2}\log|(2n/e)^{-n/2}c_n(h)| > 0$, then h has a holomorphic extension. More precisely, there exists a complex-valued function H which is holomorphic in the strip $S(\tau_0(h))$ and such that $H(x) = h(x)$ a.e. in $(-\infty, \infty)$. Moreover, for every $\tau \in [0, \tau_0(h))$ there exists a positive number $N(\tau)$ such that the inequality

$$|h(x)| \le N(\tau)\exp(x^2/2 - \tau|x|) \tag{5.8}$$

holds a.e. in $(-\infty, \infty)$.

The proof of the above proposition is similar to that of **(V.5.2)**. In particular, (5.8) is a corollary of (2.32).

5.2 A complex-valued function f, defined on a (nonempty) set $E \subset \mathbb{C}$, is called a locally Hölder function (briefly LH-function) on E if for every $\zeta \in E$ there exist a (circular) neighbourhood U of ζ, a positive constant K, and a positive number

μ (usually assumed to be less or equal to 1) such that $|f(\zeta_1)-f(\zeta_2)| \le K|\zeta_1-\zeta_2|^\mu$ whenever the points $\zeta_j \in E\cap U, j=1,2$.

Remarks. (1) U, K and μ may depend on the point $\zeta \in E$.

(2) A LH-function does not need, in general, to be a (globally) Hölder function.

We say that a complex-valued function f which is continuous on a Jordan curve $\gamma \subset \mathbb{C}$ has a holomorphic extension in the interior $G(\gamma)$ of γ if there exists a complex-valued function $F \in \mathcal{C}(\overline{G(\gamma)}) \cap \mathcal{H}(G(\gamma))$ (i.e. F is continuous on the closure of the region $G(\gamma)$ and holomorphic in $G(\gamma)$) such that $F(\zeta)=f(\zeta)$ for $\zeta \in \gamma$.

Remark. It is clear that the holomorphic extension of a given function is unique provided it exists.

The following proposition is a classical criterion for existence of holomorphic extensions (see, e.g., B.Ja.Levin, Distribution of Zeros of Entire Functions, AMS, Providence, Rhode Island, 1964, p.211. Theorem 17).

(V.5.4) *Suppose that $\gamma \subset \mathbb{C}$ is a smooth and positively oriented Jordan curve and let f be a LH-function on it. Then the following propositions are equivalent*:

(i) *f has a holomorphic extension in $G(\gamma)$;*

$$(ii)\frac{1}{\pi i}\int_\gamma \frac{f(w)}{w-\zeta}\,dw = f(\zeta),\ \zeta\in\gamma; \tag{5.9}$$

$$(iii)\int_\gamma f(w)w^n\,dw = 0,\ n=0,1,2,\dots .$$

Remark. The integral in (5.9) is understood as a principal value in the Cauchy sence, i.e.

$$\int_\gamma \frac{f(w)}{w-\zeta}\,dw := \lim_{\delta\to 0}\int_{\gamma\setminus\gamma(\zeta;\delta)} \frac{f(w)}{w-\zeta}\,dw,$$

where $\gamma(\zeta;\delta) := \gamma\cap\{w : |w-\zeta|<\delta\}$, and its existence follows from the assumption that f is a LH-function on γ.

If γ is a smooth Jordan curve in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ which passes trough the point of infinity, then, in general, the above propositions are not equivalent. Here are two examples which illustrate this.

(**I**) The function $f(x)=\exp(-x^2),\ x\in\mathbb{R}$, has a holomorphic extension both in the upper and in the lower halfplane, but

$$\int_{-\infty}^{\infty}\exp(-x^2)x^{2n}\,dx \ne 0,\ n=0,1,2,\dots .$$

Moreover, since the function $x^{-1}\exp(-x^2),\ x \in \mathbb{R}\setminus\{0\}$, is odd, we have

$$\int_{-\infty}^{\infty} \frac{\exp(-x^2)}{x}\,dx = 0,$$

i.e. in our case the singular integral equation (5.9) is not satisfied.

(**II**) Define $s(x) = x^{-\log x}\sin(2\pi\log x)$ for $x > 0$ and $s(0) = 0$. As it is well-known [I.P.Natanson, 1, p. 461: Example(Stieltjes)],

$$\int_0^{\infty} s(x)x^n\,dx = 0,\ n = 0, 1, 2, \ldots.$$

Define $f(x) = s(x)$ for $x > 0$ and $f(x) = s(-x)$ for $x \le 0$. It is clear that f is continuous on $\mathbb{R}$ and $\int_{-\infty}^{\infty} f(x)x^{2n} = 0,\ n = 0, 1, 2, \ldots$. Moreover, since f is an even function, it follows that $\displaystyle\int_{-\infty}^{\infty} f(x)x^{2n+1}\,dx = 0,\ n = 0, 1, 2, \ldots$.

If we suppose that f has a holomorphic extension F in the upper half-plane, then $F(-1) = f(-1) = s(1) = 0$. But $F(z) = \exp(-(\log z)^2)\sin(2\pi\log z)$ when $\operatorname{Im} z > 0$, hence, $\lim_{z\to -1} F(z) = \exp(\pi^2)\sin(2\pi^2 i) \neq 0$ which is a contradiction.

Remark. In each of the above examples the function f is even (global) Lipshitz function, since its derivative is bounded on the whole real axis.

5.3 Suppose that $0 < \lambda < \infty$. A Jordan curve $\gamma \subset \overline{\mathbb{C}}\setminus[0,\infty)$ passing trough the point of infinity is called λ-admissible if $\sup_{\zeta\in\gamma\setminus\{\infty\}} \operatorname{Re}(-\zeta)^{1/2} = \lambda$. It is clear that if γ is λ-admissible, then $\overline{\Delta(\lambda)}$ is the smallest closed domain which contains γ provided the closure of $\Delta(\lambda)$ is formed with respect to the extended complex plane.

Further, we denote by $G(\gamma)$ that component of $\overline{\mathbb{C}}\setminus\gamma$ which lies in $\Delta(\lambda)$ and call it the interior of γ. We suppose that γ is positively oriented with respect to $G(\gamma)$.

(V.5.5) *Suppose that $\gamma \subset \overline{\mathbb{C}}$ is a λ-admissible smooth Jordan curve and let f be a LH-function on $\gamma\setminus\{\infty\}$ such that $|f(w)| = |f(u+iv)| = O(|w|^{\beta}\exp(-u))$ for some $\beta \in \mathbb{R}$ when $w = u + iv$ tends to infinity. If*

$$\int_{\gamma} f(w)w^n\,dw = 0,\ n = 0, 1, 2, \ldots, \tag{5.10}$$

then f has a holomorphic extension into the region $G(\gamma)$.

Proof. Define

$$F_{\gamma}(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(w)}{w-z}\,dw,\ z \in G(\gamma)$$

and

$$F_{\gamma}^{*}(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(w)}{w-z}\,dw,\ z \in G^{*}(\gamma) := \mathbb{C}\setminus\overline{G(\gamma)}.$$

It is easy to prove that for each $\zeta \in \gamma \setminus \{\infty\}$ there exist the limits

$$\lim_{z\to\zeta} F_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-\zeta}\,dw + \frac{1}{2}f(\zeta), \tag{5.11}$$

$$\lim_{z\to\zeta} F^*_\gamma(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-\zeta}\,dw - \frac{1}{2}f(\zeta). \tag{5.12}$$

Indeed, there exists $\delta > 0$ such that $\gamma(\zeta;\delta)$ is a smooth Jordan arc and, moreover, the function f satisfies a Hölder condition on $\gamma(\zeta)$. If we define

$$\Phi_\delta(f;z) = \frac{1}{2\pi i}\int_{\gamma(\zeta;\delta)} \frac{f(w)}{w-z}\,dw,\ \ z \in \mathbb{C}\setminus\gamma(\zeta;\delta), \tag{5.13}$$

and

$$\Psi_\delta(f;z) = \frac{1}{2\pi i}\int_{\gamma\setminus\gamma(\zeta;\delta)} \frac{f(w)}{w-z}\,dw,\ \ z \in \mathbb{C}\setminus(\gamma\setminus\gamma(\zeta;\delta)), \tag{5.14}$$

then there exist

$$\lim_{z\in G(\gamma), z\to\zeta} \Phi_(\delta)(f;z) = \frac{1}{2\pi i}\int_{\gamma(\zeta;\delta)} \frac{f(w)}{w-\zeta}\,dw + \frac{1}{2}f(\zeta), \tag{5.15}$$

$$\lim_{z\in G^*(\gamma), z\to\zeta} \Phi_\delta \frac{1}{2\pi i}\int_{\gamma(\zeta;\delta)} \frac{f(w)}{w-\zeta}\,dw - \frac{1}{2}f(\zeta) \tag{5.16}$$

and

$$\lim_{z\in\mathbb{C}\setminus(\gamma\setminus\gamma(\zeta;\delta))} \Psi_\delta(f;z) = \frac{1}{2\pi i}\int_{\gamma\setminus\gamma(\zeta;\delta)} \frac{f(w)}{w-\zeta}\,dw. \tag{5.17}$$

Since $\Phi_\delta(f;z)+\Psi_\delta(f;z) = F_\gamma(z)$ for $z \in G(\gamma)$ and $\Phi_\delta(f;z)+\Psi_\delta(f;z) = F^*_\gamma(z)$ for $z \in G^*(\gamma)$, the equalities (5.11) and (5.12) follow from (5.13), (5.14), (5.15), (5.16) and (5.17).

If we choose a real $\alpha > 2\beta - 1/2$, then $\beta < \alpha/2 + 1/4$. Hence, we can assert that the representation

$$F^*_\gamma(z) = \sum_{n=0}^{\infty} b_n M_n^{(\alpha)}(z)$$

holds for $z \in \Delta^*(\lambda)$ and, moreover, that the coefficients $\{b_n\}_{n=0}^{\infty}$ are given by the equalities which we get from (4.12) replacing μ_0 by λ.

It was already mentioned that the system of Laguerre polynomials is a basis in the space of (algebraic) polynomials. Then from the assumption (5.10) it follows that $b_n = 0$ for $n = 0, 1, 2, \ldots$, i.e. that the function F_γ^* is identically zero in the region $\Delta^*(\lambda)$. Since $G^*(\lambda) \supset \Delta^*(\lambda)$, the identity theorem implies that $F_\gamma^*(z) = 0$ for $z \in G^*(\lambda)$. Further, (5.12) yields

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-\zeta}\,dw = \frac{1}{2}f(\zeta) \tag{5.18}$$

for $\zeta \in \gamma$. Hence, if we define $F_\gamma(\zeta) = f(\zeta)$ for $\zeta \in \gamma$, then F_γ is the holomorphic extension of the function f into the region $G(\lambda)$. Indeed, due to (5.11), we have $\lim_{z\in G(\lambda), z\to\zeta} F_\gamma(z) = f(\zeta)$ for $\zeta \in \gamma$.

Remark. From (5.18) it follows that the function f satisfies the singular integral equation (5.9).

It seems that, in general, the converse of proposition **(V.5.5)** is not true but under some additional assumption it can be "reversed".

We say that a λ-admissible Jordan curve $\gamma \subset \overline{\mathbb{C}}$ is regular at infinity if the intersection $C(\gamma;\rho) := \overline{G(\lambda)} \cap C(0;\rho)$, where $C(0;\rho)$ is the circle with center at the origin and radius ρ, has only one component, provided $\rho \geq \rho_0$ and ρ_0 is large enough.

(V.5.6) *Suppose that $\gamma \subset \overline{\mathbb{C}}$ is a λ-admissible smooth Jordan curve regular at infinity. Let the LH-function $f : \gamma \setminus \{\infty\} \longrightarrow \mathbb{C}$ satisfy the growth condition in* **(V.5.5)**. *If f has a holomorphic extension F_γ into the region $G(\lambda)$ such that $|F_\gamma(z)| = O(|z|^\omega)$ for some $\omega < 1/2$ when z tends to infinity, then the equalities* (5.10) *hold.*

Proof. If $\rho \geq \rho_0$ and $\rho_0 > 2\lambda^2$ is large enough, then the Cauchy theorem implies that for $z \in G^*(\lambda)$,

$$\int_\gamma \frac{F_\gamma(w)}{w-z}\,dw = \int_{\gamma_\rho} \frac{f(w)}{w-z}\,dw + \int_{C(\gamma;\rho)} \frac{F_\gamma(w)}{w-z}\,dw = 0, \tag{5.19}$$

where $\gamma_\rho := \gamma \cap \{w : |w| \leq \rho\}$.

Denote by $\zeta^*(\lambda,\rho)$ those of the endpoints of the circular arc $C(\lambda;\rho) := \overline{\Delta(\lambda)} \cap C(0;\rho)$ for which $\operatorname{Im}\zeta^*(\lambda;\rho) > 0$. If $\theta^*(\lambda;\rho) = \arg\zeta^*(\lambda;\rho)$, then

$$\tan\theta^*(\lambda;\rho) = 2\lambda(\rho-\lambda^2)^{1/2}(\rho-2\lambda^2)^{-1}.$$

Further, for the length $l(\gamma;\rho)$ of $C(\gamma;\rho)$ we obtain that

$$l(\gamma;\rho) \leq 2\rho\theta^*(\lambda;\rho) = 2\rho\arctan(2\lambda(\rho-\lambda^2)^{1/2}(\rho-2\lambda^2)^{-1}),$$

and, hence, $l(\gamma;\rho) = O(\rho^{1/2})$ when ρ tends to infinity. Further,

$$\left|\int_{C(\gamma;\rho)} \frac{F_\gamma(w)}{w-z}\,dw\right| \leq \int_{C(\gamma;\rho)} \frac{|F_\gamma(w)|}{|w-z|}\,ds \leq \mathrm{Const}(F_\gamma, z)\rho^{\omega-1/2},$$

and letting $\rho \to \infty$, from (5.19) we obtain that $F_\gamma^*(z) = 0$. Since z is arbitrary, it follows that $F_\gamma^* \equiv 0$ in the region $G^*(\gamma)$, i.e. $\sum_{n=0}^{\infty} b_n M_n^{(\alpha)}(z) = 0$ for $z \in \Delta^*(\lambda)$. Then the uniqueness property of the expansions in series of Laguerre associated functions [**(IV.5.10)**] yields that $b_n = 0,\ n = 0, 1, 2, \dots$, i.e.

$$\int_\gamma f(w) L_n^{(\alpha)}(w)\, dw = 0,\ n = 0, 1, 2, \dots,$$

Since the system of Laguerre polynomials is a basis in the space of polynomials, the equalities (5.10) follow.

5.4 Suppose that $0 < \tau < \infty$. A Jordan curve $\gamma \subset \overline{\mathbb{C}}$ passing trough the point of infinity is called $\tau^+(\tau^-)$-admissible if $-\infty < \inf_{\zeta \in \gamma \setminus \{\infty\}} \operatorname{Im} \zeta \le \sup_{\zeta \in \gamma \setminus \{\infty\}} \operatorname{Im} \zeta = -\tau(\tau = \inf_{\zeta \in \gamma \setminus \{\infty\}} \operatorname{Im} \zeta \le \sup_{\zeta \in \gamma \setminus \{\infty\}} \operatorname{Im} \zeta < \infty)$.

By $H^+(\gamma)(H^-(\gamma))$ we denote the component of $\mathbb{C} \setminus \gamma$ which contains the real axis and suppose γ is positively orientiated with respect to $H^+(\gamma)(H^-(\gamma))$.

Using series representations by the Hermite associated functions $\{G_n^\pm(z)\}_{n=0}^\infty$ as well as the uniqueness of the expansions in these functions we can prove the following propositions:

(V.5.7) *Suppose that γ is a $\tau^+(\tau^-)$-admissible smooth Jordan curve and let f be a LH-function on $\gamma \setminus \{\infty\}$ such that*

$$|f(w)| = |f(u + iv)| = O(|w|^{-\sigma} \exp(-u^2)) \tag{5.20}$$

for some $\sigma > 0$ when w tends to infinity. If the equalities (5.10) *hold, then f has a holomorphic extension into the region $H^+(\gamma)(H^-(\gamma))$.*

(V.5.8) *Suppose that $\gamma \subset \overline{\mathbb{C}}$ is a $\tau^+(\tau^-)$-admissible smooth Jordan curve which is regular at infinity and let the LH-function f on $\gamma \setminus \{\infty\}$ satisfy the growth condition (5.20). If f has a holomorphic extension $F_\gamma^+(F_\gamma^-)$ into the region $H^+(\gamma)(H^-(\gamma))$ such that $|F_\gamma^+(z)| = o(1)(|F_\gamma^-(z)| = o(1))$ when $z \in H^+(\gamma)(H^-(\gamma))$ tends to infinity, then the equalities* (5.10) *hold.*

Exercises

1. Prove **(V.1.3)**.

2. Prove that each complex function which is holomorphic in a region of the kind $E^*(r) \cap E(R), 1 \le r < R \le \infty$, has there unique representation by a series of the kind

$$\sum_{n=0}^{\infty} \{a_n P_n^{(\alpha,\beta)}(z) + b_n Q_n^{(\alpha,\beta)}(z)\}.$$

3. Prove that if $\operatorname{Re}\zeta < 1/2$, then for $z \in \mathbb{C}$,

$$z\exp(\zeta z^2) = \sum_{n=0}^{\infty} \frac{(1-\zeta)^{-1/2}}{n!2^{2n+1}} \left\{\frac{\zeta}{1-\zeta}\right\}^{n+1} H_{2n+1}(z).$$

4. Prove that if $|\zeta| < 1$, then for $z \in \mathbb{C}$:

$$(i)\ (1+\zeta)^{-a}\Phi(a, 1/2; z^2\zeta(1+\zeta)^{-1}) = \sum_{n=0}^{\infty} \frac{(a)_n\zeta^n}{(2n)!} H_{2n}(z);$$

$$(ii)\ z(1+\zeta)^{-a}\Phi(a, 3/2; z^2\zeta(1+\zeta)^{-1}) = \sum_{n=0}^{\infty} \frac{(a)_n\zeta^n}{(2n+1)!} H_{2n+1}(z),$$

where a is an arbitrary complex number and $\Phi(a, c; z)$ is the Kummer degenerate hypergeometric function [Appendix , (3.8)].

5. Prove that the representation

$$(1+\zeta)^{-a}\Phi(a, \alpha+1; z\zeta(1+\zeta)^{-1}) = \sum_{n=0}^{\infty} (-1)^n \frac{(a)_n\zeta^n}{(\alpha+1)_n} L_n^{(\alpha)}(z)$$

holds for $z \in \mathbb{C}$ provided that $\alpha \neq -1, -2, -3, \dots$ and $|\zeta| < 1$.

6. Prove **(V.4.5)**.

7. Suppose that $0 < \tau < \infty$ and let F be a locally L-integrable complex function on the line $l(\pm\tau) : \zeta = t \pm i\tau, -\infty < t < \infty$ such that $|F(\zeta)| = O(|\zeta|^{-\mu}\exp(-\xi^2))$ for some $\mu > 0$ when $\zeta = \xi + i\eta$ tends to infinity. Prove that for $z \in H^{\pm}(\pm\tau)$ the function

$$f(z) = \pm(2\pi i)^{-1} \int_{l(\pm\tau)} \frac{F(\zeta)}{\zeta - z}\, d\zeta,\ z \in \mathbb{C} \setminus l(\pm\tau)$$

is representable by a series in the Hermite associated functions $\{G_n^{\pm}(z)\}_{n=0}^{\infty}$ with coefficients

$$b_n = \mp(2\pi i I_n)^{-1} \int_{l(\pm\tau)} H_n(\zeta)F(\zeta)\, d\zeta,\ n = 0, 1, 2, \dots.$$

8. Suppose that $0 < \lambda < \infty, \alpha \in \mathbb{R} \setminus \mathbb{Z}^-$ and let F be a L-integrable complex function on the ray $(-\infty, -\lambda_0^2]$. If $\displaystyle\int_{-\infty}^{-\lambda_0^2} |F(t)||t|^{\sigma(\alpha)}\, dt < \infty$, where $\sigma(\alpha) = \max(-1, \alpha/2 - 5/4)$. Prove that the equalities

$$\int_{-\infty}^{-\lambda_0^2} M_n^{(\alpha)}(t)F(t)\, dt = 0,\ n = 0, 1, 2, \dots,$$

imply $F \sim 0$, i.e. that F is almost everywhere equal to zero in the interval $(-\infty, -\lambda_0^2]$.

9. Suppose that $0 < \tau_0 < \infty$ and let $\{G_n^{(k)}(z;\tau_0)\}_{n=0}^{\infty}, k = 1, 2$ be the functions defined in [III, Exercise. 4]. Suppose that the complex function f is holomorphic in the strip $S(\tau_0)$ and that for each $\tau \in [0, \tau_0)$ there exists $\mu(\tau) > 0$ such that $|f(z)| = \{|z|^{-\mu(\tau)} \exp(-x^2)\}$ when $z = x + iy$ tends to infinity in $\overline{S}(\tau)$. Prove that the function f has the representation

$$f(z) = \sum_{n=0}^{\infty}\{a_n^{(1)}(f)G_n^{(1)}(z;\tau_0) + a_n^{(2)}(f)G_n^{(2)}(z;\tau_0)\},\ z \in S(\tau_0)$$

with coefficients $(n = 0, 1, 2, \dots)$

$$a_n^{(1)}(f) = \frac{1}{4\pi i I_n}\int_{-\infty}^{\infty}\{H_n(t + i\tau_0) + (-1)^n H_n(t - i\tau_0)\}f(t)\,dt;$$

$$a_n^{(2)}(f) = -\frac{1}{4\pi i I_n}\int_{-\infty}^{\infty}\{H_n(t + i\tau_0) - (-1)^n H_n(t - i\tau_0)\}f(t)\,dt.$$

10. We say that a function $f \in \mathcal{H}(S(\tau_0)), 0 < \tau_0 \le \infty$, has Cauchy's integral representation in the strip $S(\tau_0)$ if for any $\tau \in (0, \tau_0)$ and $z \in S(\tau)$

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left\{\frac{f(t - i\tau)}{t - i\tau - z} - \frac{f(t + i\tau)}{t + i\tau - z}\right\}dt.$$

Prove that if, in addition, $\displaystyle\int_{-\infty}^{\infty}|f(t \pm i\tau)|\,dt < \infty$ for $\tau \in (0, \tau_0)$, then in order that f has an expansion for $z \in S(\tau_0)$ in a series of Hermite polynomials with coefficients (2.42) it is necessary and sufficient f to have Cauchy type integral representation in $S(\tau_0)$.

Comments and references

From "complex analytic" point of view one of the main problems concerning the Jacobi, Laguerre and Hermite systems is that of representing holomorphic functions by series in these systems.

To solve the representation problem for a given region $G \subset \overline{\mathbb{C}}$ and a sequence $F = \{f_n\}_{n=0}^{\infty}$ of functions holomorphic in G means to describe the ($\mathbb{C}$-vector) subspace $\mathcal{F}(G; F)$ of $\mathcal{H}(G)$ consisting of all functions which are holomorphic in G and have there expansions in series of the kind $\displaystyle\sum_{n=0}^{\infty} a_n f_n(z)$ which converge in the sence of the topology of the space $\mathcal{H}(G)$. If, in addition, the system F has the uniqueness property, then it is simply a basis of the space $\mathcal{F}(G; F)$.

Let us mention that, in general, $\mathcal{F}(G;F)$ does not coincide with the whole space $\mathcal{H}(G)$. It may happen also that $\mathcal{F}(G;F)$ is not closed in $\mathcal{H}(G)$, i.e. that it may be not a subspace of $\mathcal{H}(G)$ as a Fréchet space.

If the simply connected region $G \subset \mathbb{C}$ is unbounded and, moreover, the functions $f_n,\ n = 0, 1, 2, \ldots,$ have no finite singular points (in particular if f_n is a polynomial of degree n), then usually we are trying to find "growth" characteristic of the functions in the space $\mathcal{F}(G;F)$.

In the case of Jacobi systems we need no growth conditions when proving assertions about the representation of holomorphic functions by series in these systems. The "heavy" role is played by the asymptotic formulas for the Jacobi polynomials and their associated functions. In this way we obtain e.g. the proposition **(V.1.2)**, and by the use of the relation [Chapter I, Exercise. 4] we come to **(V.1.3)**. Let us mention that in [B.C.Carlson, 2] the expansions of holomorphic functions in series of Jacobi polynomials are regarded as generalized Taylor's series rather than as series in special orthogonal polynomials.

The possibility to make use of the relation [Chapter I, Exercise 4] when the representation by series in Jacobi polynomials is discussed has been noticed by Boas and Buck [1, p. 61] and developed in details by D. Colton [1]. In the last paper it is proved also that, in general, the expansion in a series of Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ with real α and β is not possible, when $\alpha + \beta + 2$ is a nonpositive integer.

The fact that a complex function holomorphic in the interior of an ellipse with foci at the points ± 1 is representable there by a series in Jacobi polynomials has been established in the 19th century. The corresponding proposition is formulated in [G. Szegö, Theorem 9.1.1] without giving any comments and historical references. For the expansion (9.2.1) from [G. Szegö, 1], which is nothing but a bilinear representation of the Cauchy kernel in terms of the Jacobi polynomials and their associated functions, it is said that it is well-known in the special case $\alpha = \beta = 0$ (E. Heine [1, p. 78]).

Let us note that (9.2.1) as well as Theorem 9.1.1 and Theorem 9.2.2 from [G. Szegö, 1] are due to G. Darboux [1]. In fact, Darboux has proved the validity of the proposition given as Exercise 2.

A proof of **(V.1.3)** is given [D. Colton, 1] provided $\alpha \le -1, \beta \le -1$ and $\alpha + \beta + 2 \ne 0, -1, -2, \ldots$. In the same paper it is proved that this proposition is not valid if α, β are not equal to $-1, -2, -3, \ldots$ but $\alpha+\beta+2 = 0, -1, -2, \ldots$. More precisely, it is shown that if $-\alpha - \beta - 1$ is a (positive) integer, then a polynomial of degree $-\alpha - \beta - 1$ cannot be expanded in a series of the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$

The solution of representation problem in the case of Hermite and Laguerre polynomials has a longer history. For expansions in the Hermite functions of complex-valued functions, which are holomorphic in a strip of the kind $S(\tau_0)$, G. Szegö [1, p. 253] refers to the first part of the paper [G.N. Watson, 1]. Sufficient conditions for existence of series representations of holomorphic functions by means of Laguerre and Hermite polynomials are given also by O. Volk [1].

In 1940 appeared the fundamental paper of E. Hille [1], where the representation problem by Hermite polynomials is solved. The essential part of this paper is represented in Section **2**. The only difference is that we use the completeness of Hermite functions in the space $L^2(\mathbb{R})$ [**(V.2.1)**] instead of Abel's summability of the series representation by this system of a complex measurable function satisfying suitable "growth" condition on the real line.

The main tools used by Hille is the bilinear generating function [Chapter II, (2.6)] for the Hermite polynomials as well as the "asymptotic" representation [V, (2.15)] of the Weber-Hermite functions obtained as an application of results due to Langer [1]. For the details we refer to Hille's paper [1].

The main result in Section **3** is the proposition **(V.3.7)**. It describes the growth of the functions which are holomorphic in a region of the kind $\Delta(\lambda_0), 0 < \lambda \leq \infty$ and have there expansions in series of Laguerre polynomials.

The solution of the representation problem in the case of the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ for $\alpha = 0$ was given by H. Pollard [1]. The validity of Pollard's theorem when $\alpha > -1$ was proved by O. Százs and N. Yeardley [1]. By establishing the sufficiency of the growth condition (1.5) in their paper they make use of the equality [Lemma 3.2]

$$\int_x^{\infty} (u-x)^{-\delta} \exp(-u) L_n^{\alpha}(u)\, du = \Gamma(1-\delta) \exp(-x) L_n^{\alpha+\delta-1}(x),$$

which holds when $\alpha > -1, \delta < 1$ and $x \geq 0$. Let us note that its particular case $\alpha = \delta = 1/2$ has been already used by Pollard [1,(3.2)].

Our approach is based on Uspensky's formula [Chapter II, (2.17)] and on the proposition **(V.3.3)** [P. Rusev,5,6]. As a corollary we obtain a generalization of Pollard's theorem as well as of that of Százs and Yeardley.

The proposition **(V.2.10)** is due to P. Rusev [18, Theorem 3]. The propositions **(V.3.10), (V.3.14), (V.3.16)** and **(V.4.3)** were announced in [P. Rusev, 15] and their proofs appeared in [P. Rusev,16]. The proof of **(V.3.16)** is included also in [P. Rusev, 17].

The proofs of **(V.3.11)** and **(V.3.12)** are published in [P. Rusev, 21] and [P. Rusev, 22], respectively. The propositions **(V.3.15)** and **(V.4.2)** are included in [P. Rusev, 19, pp. 191, 202].

The proposition **(V.4.5)** is published in [P. Rusev, Theorem 5]. The assertions given as Exercises **8** and **9** are included in [P. Rusev, 15] and [P. Rusev, 20], respectively, and that as Exercise **10** is also due to P. Rusev.

The results concerning the holomorphic extension of measuarable function of one (real) variable are published in [P. Rusev, 34] and these for the holomorphic extension of locally Hölder functions which are defined on Jordan curves in the extended complex plane are included in [P. Rusev, 35].

Chapter VI

THE REPRESENTATION PROBLEM IN TERMS OF CLASSICAL INTEGRAL TRANSFORMS

1. Hankel transform and the representation by series in Laguerre polynomials

1.1 The integral representation [Chapter II, (2.13)] of the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$, i.e.

$$L_n^{(\alpha)}(z) = \frac{\exp z}{n!} \int_0^\infty \exp(-t/2) B_\alpha(zt)\, dt,$$

shows that the entire functions $\exp(-z)L_n^{(\alpha)}(z)$ is a Hankel type integral transform of the entire function $(n!)^{-1}z^n \exp(-z)$ provided $\operatorname{Re}(n+\alpha+1) > 0$. This makes plausible the conjecture that if a complex function $f \in \mathcal{H}(\Delta(\lambda_0))$, $0 < \lambda_0 \le \infty$, is representable by a series in Laguerre polynomials in the region $\Delta(\lambda_0)$, then the function $\exp(-z)f(z)$ has to be a Hankel transform of a suitable entire function.

In order to justify this conjecture we introduce the class $G(\sigma)$, $-\infty < \sigma \le \infty$, of entire functions Φ such that:

$$\limsup_{|w|\to\infty} (2\sqrt{|w|})^{-1}(\log|\Phi(w)| - |w|) \le \sigma. \tag{1.1}$$

The functions in $G(\sigma)$ can be described by means of a growth condition concerning the coefficients of their Taylor series expansions centered at the origin. The following lemma plays an auxiliary role.

(VI.1.1) *If* $\operatorname{Re}\alpha > -1$, *then there exists a positive constant* $M(\alpha)$ *such that the inequality*

$$|B_\alpha(z)| \le M(\alpha)(1+|z|)\exp(2|\operatorname{Im}\sqrt{z}|) \tag{1.2}$$

holds for $z \in \mathbb{C}$.

Proof. The relation $B_\alpha(z) + zB_{\alpha+2}(z) = (\alpha+1)B_{\alpha+1}(z)$, which follows from [Appendix, (2.8)], and the inequality [Chapter III, (4.7)] yield for $z \in \mathbb{C}$

$$|B_\alpha(z)| \le (K(\alpha+1)|\alpha+1| + K(\alpha+2)|z|)\exp(2|\operatorname{Im}\sqrt{z}|).$$

Denoting $M(\alpha) = \max\{K(\alpha+1)|\alpha+1|, K(\alpha+2)\}$, we obtain (1.2).

(VI.1.2) *An entire function*

$$\Phi(w) = \sum_{n=0}^{\infty} (n!)^{-1} a_n w^n \tag{1.3}$$

belongs to the class $G(\sigma)$ if and only if

$$\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n| \le -\sigma. \tag{1.4}$$

Proof. Suppose that $\sigma < \infty$ and that the entire function (1.3) is in the class $G(\sigma)$. If $\varepsilon > 0$, then there exists a positive integer $n_0 = n_0(\varepsilon)$ such that

$$|a_n| \le n!n^{-n}\max_{|w|=n}|\Phi(w)| \le n!n^{-n}\exp[-n - 2(\sigma-\varepsilon)\sqrt{n}],\ n > n_0.$$

Stirling's formula yields that $\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n| \le -\sigma+\varepsilon$, and since $\varepsilon > 0$ is arbitrary, we come to the inequality (1.4).

If $\sigma = \infty$, then $|a_n| \le n!n^{-n}\exp(n - 2A\sqrt{n})$ for $A > 0$ and sufficiently large n and, hence, $\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n| \le -A$. Since $A > 0$ is arbitrary, we obtain that $\lim_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n| = -\infty$.

In order to prove the sufficiency of (1.4) we shall consider separately the cases $\sigma > 0$ and $\sigma \le 0$.

(a) Suppose that $0 < \sigma < \infty$. Since the sequence $\{a_n\}_{n=0}^{\infty}$ satisfies (1.4), then from the asymptotic formula [III, (3.11)] it follows that for $\delta \in (0,\sigma)$ there exists a constant $C(\delta) > 0$ such that $|a_n| \le C(\delta)\{-M_n^{(0)}(-(\sigma-\delta)^2)\}$ for $n = 0,1,2,\ldots$. Then [II, (4.5)] yields that $|\Phi(w)| \le C(\delta)M^{(0)}(-(\sigma-\delta)^2,|w|)$ and [II, (4.4)] gives

$$|\Phi(w)| = O\Big\{\int_0^\infty \exp[|w|t(1+t)^{-1} - (\sigma-\delta)^2t]\,dt\Big\}$$

$$= O\Big\{\int_0^\infty \exp[|w| - |w|(1+t)^{-1} - (\sigma-\delta)^2t]\,dt\Big\}$$

$$= O\Big\{\int_0^\infty \exp[|w| - (\sigma-\delta)^2t - |w|t^{-1}]\,dt\Big\}$$

$$= O\Big\{\sqrt{|w|}\int_{(\sigma-\delta)/\sqrt{|w|}}^\infty \exp[|w| - (\sigma-\delta)\sqrt{|w|}(u+u^{-1})]\,du\Big\}.$$

If $|w|$ is large enough, then we write the integral as a sum of two integrals namely: one of them on the segment $[(\sigma-\delta)/\sqrt{|w|},1]$, and the other on the ray $[1,\infty)$. Further, we substitute $u = (\tau+\sqrt{\tau^2-4})/2,\ 2 \le \tau < \infty$ in both of them and obtain that

$$|\Phi(w)| = O\Big\{\sqrt{|w|}\int_2^\infty [1+\tau(\tau^2-4)^{-1/2}]\exp[|w| - (\sigma-\delta)\sqrt{|w|}\tau]\,d\tau\Big\}. \tag{1.5}$$

It is easy to prove that if $\lambda > 0$, and if $a(\tau)$ is a continuous function of τ

$\in (\lambda, \infty)$ such that $a(\tau) = O((\tau - \lambda)^{-\eta})$ for some $\eta < 1$ when $\tau \to \lambda$, and $a(\tau) = O(1)$ when $\tau \to \infty$, then

$$\int_{\lambda}^{\infty} a(\tau) \exp(-r\tau)\, d\tau = O(\exp(-\lambda r)) \tag{1.6}$$

for $r \to \infty$.

Then (1.6) and (1.5) yield that $|\Phi(w)| = O\{\sqrt{|w|}exp[|w| - 2(\sigma - \delta)\sqrt{|w|}]\}$ and, hence, $\limsup_{|w|\to\infty}(2\sqrt{|w|})^{-1}(\log|\Phi(w)| - |w|) \leq -\sigma + \delta$. Since $\delta \in (0, \sigma)$ is arbitrary, we come to the conclusion that the entire function Φ belongs to the class $G(\sigma)$. The proof in the case $\sigma = \infty$ is similar to the above one and we omit it.

(b) Suppose that $-\infty < \sigma \leq 0$. Then, having in mind the asymptotic formula [Chapter III, (2.3)] for the Laguerre polynomials, we can assert that for every $\delta > 0$ there exists a positive constant $D(\delta)$ such that the inequality

$$|a_n| \leq D(\delta) L_n^{(0)}(-(\sigma - \delta)^2) \tag{1.7}$$

holds for $n = 0, 1, 2, \ldots$.

Remark. Notice that the validity of the above inequality follows from the fact that $L_n^{(0)}(-x) > 0$ for $x > 0$.

Further, with the aid of [Chapter II, (2.7)], we find that

$$|\Phi(w)| = \left\{\sum_{n=0}^{\infty}(n!)^{-1}L_n^{(0)}(-(\sigma-\delta)^2)|w|^n\right\} = O\{\exp(|w|)B_0(-(\sigma-\delta)^2)|w|\},$$

and then the inequality (1.2) yields that

$$|\Phi(w)| = O\{|w|\exp[|w| + 2(-\sigma + \delta)\sqrt{|w|}]\}.$$

Since $\delta \in (0, \sigma)$ is arbitrary, the above estimate implies that $\Phi \in G(\sigma)$.

Remark. It is clear that $G(\sigma), \sigma \in (-\infty, \infty]$ is a $\mathbb{C}$-vector space.

1.2 Suppose that $0 < \lambda_0 \leq \infty$ and $\alpha \in \mathbb{C}$. Define the linear map $H_{\lambda_0,\alpha}$ of the space $G(\lambda_0)$ in the space $\mathcal{L}^{(\alpha)}(\lambda_0)$ assuming that to a function $\Phi \in G(\lambda_0)$ having the representation (1.3) there corresponds the function $f \in \mathcal{L}^{(\alpha)}(\lambda_0)$ which is defined by the series [IV, (2.1)].

The linearity of $H_{\lambda_0,\alpha}$ follows immediately from the definition. It is easy to prove that $H_{\lambda_0,\alpha}$ is surjective. Indeed, if a function $f \in \mathcal{L}^{(\alpha)}(\lambda_0)$ is represented in $\Delta(\lambda_0)$ by a series of the kind [ChapterIV,(2.1)], then from **(IV.2.1)** $\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n| \leq -\lambda_0$. Due to **(VI.1.2)**, the entire function Φ defined by (1.3) is in the space $G(\lambda_0)$ and, moreover, $f = H_{\lambda_0,\alpha}(\Phi)$.

(VI.1.3) *Suppose that* $0 < \lambda_0 \leq \infty$ *and* $\operatorname{Re}\alpha > -1$. *Then* $f \in H_{\lambda,\alpha}(\Phi)$ *if and only if it has the form*

$$f(z) = \exp z \int_0^\infty t^\alpha \exp(-t)\Phi(t)B_\alpha(zt)\,dt,\ z \in \Delta(\lambda_0). \tag{1.8}$$

In other words, a complex-valued function f *is in the space* $\mathcal{L}^{(\alpha)}(\lambda_0)$ *if and only if the above representation holds with a function* $\Phi \in G(\lambda_0)$.

Proof. If $\Phi \in G(\lambda_0)$, then the integral in the right-hand side of (1.8) is absolutely uniformly convergent on every compact subset K of the region $\Delta(\lambda_0)$. Suppose that $\lambda_0 < \infty$ and define $\lambda = \max\{|\operatorname{Im}\sqrt{z}| : z \in K\}$, and $r = \max\{|z| : z \in K\}$. It is clear that $0 \leq \lambda < \lambda_0$ and $0 \leq r < \infty$. If $\delta = (\lambda_0 - \lambda)/2$, then the definition of the class $G(\sigma)$ as well as the inequality (1.2) yields

$$|t^\alpha \exp(-t)\Phi(t)B_\alpha(zt)| = O\{t^{\operatorname{Re}\alpha}(1+rt)\exp[-2(\lambda_0-\delta)\sqrt{t} + 2\lambda\sqrt{t}]\}$$

$$= O\{t^{\operatorname{Re}\alpha}(1+rt)\exp(-2\delta\sqrt{t})\}$$

provided $z \in K$ and $t \to \infty$.

In other words, the integral

$$\int_0^\infty |t^\alpha \exp(-t)\Phi(t)B_\alpha(zt)|\,dt$$

is majorizied on the set K by the integral

$$\int_0^\infty t^{\operatorname{Re}\alpha}(1+rt)\exp(-2\delta\sqrt{t})\,dt$$

and, hence, it is uniformly convergent on K. In a similar way, we consider the case $\lambda_0 = \infty$.

Suppose that the complex-valued function $f \in \mathcal{H}(\Delta(\lambda_0))$ has in $\Delta(\lambda_0)$ a representation of the kind (1.8), where $\Phi \in G(\lambda_0)$. If (1.3) is the Taylor expansion of Φ centered at the origin, then the inequality (1.4) holds with $\sigma = \lambda_0$, and from **(IV.2.1)** it follows that the series in Laguerre's polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^\infty$ with coefficients $\{a_n\}_{n=0}^\infty$ converges in the region $\Delta(\lambda_0)$.

If we define $R_\nu(z) = f(z) - \sum_{n=0}^{\nu} a_n L_n^{(\alpha)}(z)$ for $z \in \Delta(\lambda_0)$ and $\nu = 0,1,2,\dots$, then from [Chapter II, (2.13)], (1.3) and (1.8) it follows that

$$R_\nu(z) = \exp z \int_0^\infty t^\alpha \exp(-t)\left\{\sum_{n=\nu+1}^\infty (n!)^{-1}a_n t^n\right\}B_\alpha(zt)\,dt. \tag{1.9}$$

Further, using **(VI.1.2)**, we conclude that the entire function

$$\Phi^*(w) = \sum_{n=0}^{\infty}(n!)^{-1}|a_n|w^n$$

is in the class $G(\lambda_0)$. Therefore, for $z \in \Delta(\lambda_0)$ and $\varepsilon > 0$ there exists a $T = T(z,\varepsilon) > 0$ such that

$$\int_T^{\infty} t^{\operatorname{Re}\alpha} \exp(-t)\Phi^*(t)|B_\alpha(zt)\,dt < \varepsilon.$$

Then for $\nu = 0,1,2,\dots$ we have

$$\left|\int_T^{\infty} t^{\alpha}\exp(-t)\left\{\sum_{n=\nu+1}^{\infty}(n!)^{-1}a_n t^n\right\}B_\alpha(zt)\,dt\right| \tag{1.10}$$

$$\leq \int_T^{\infty} t^{\operatorname{Re}\alpha}\exp(-t)\Phi^*(t)|B_\alpha(zt)|\,dt < \varepsilon.$$

There exists a nonnegative integer $N = N(\varepsilon,T)$ such that for $\nu \geq N$ and $t \in [0,T]$, $\left|\sum_{n=\nu+1}^{\infty}(n!)^{-1}a_n t^n\right| < \varepsilon$ and, hence,

$$\left|\int_0^{T} t^{\alpha}\exp(-t)\left\{\sum_{n=\nu+1}^{\infty}(n!)-1a_n t^n\right\}B_\alpha(zt)\,dt\right| \tag{1.11}$$

$$= O\left\{\varepsilon\int_0^{T} t^{\operatorname{Re}\alpha}\exp(-t)|B_\alpha(zt)|\,dt\right\} = O(\varepsilon).$$

From (1.9), (1.10) and (1.11) it follows that $R_\nu(z) = O(\varepsilon)$, $\nu \geq N$, i.e. the series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ with coefficients $\{a_n\}_{n=0}^{\infty}$ represents the function f in the region $\Delta(\lambda_0)$.

Conversely, let the complex function $f \in H(\Delta(\lambda_0)), 0 < \lambda_0 \leq \infty$, be represented for $z \in \Delta(\lambda_0)$ by a series in the polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ provided $\operatorname{Re}\alpha > -1$. If $\{a_n\}_{n=0}^{\infty}$ are the coefficients of this series, then from **(IV.2.1)** and **(VI.1.2)** it follows that the entire function Φ, defined by (1.3), is in the class $G(\lambda_0)$. Then the function $\tilde{f}$ defined in the region $\Delta(\lambda_0)$ by the right-hand side of (1.8) is holomorphic in this region. But, as we have just seen, the function $\tilde{f}$ is representable in $\Delta(\lambda_0)$ by a series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ with the same coefficients $\{a_n\}_{n=0}^{\infty}$. It means that $\tilde{f} \equiv f$, i.e. the function f has the representation (1.8) in the region $\Delta(\lambda_0)$.

Example. Every entire function of exponential type and type less than one is in the class $G(\infty)$. In particular, the function $f(z) = \exp[-\zeta(1-\zeta)^{-1}z]$ is in

$G(\infty)$ when $\operatorname{Re}\zeta < 1/2$. Then from **(VI.1.3)** we obtain for $z \in \mathbb{C}$,

$$\int_0^\infty t^\alpha \exp(-(1-\zeta)^{-1}t) B_\alpha(zt)\, dt = (1-\zeta)^{1+\alpha} \exp(-(1-\zeta)z), \tag{1.12}$$

provided $\operatorname{Re}\zeta < 1/2$.

1.3 Now we are going to prove that the map $H_{\lambda_0,\alpha}$ for $\alpha \in \mathbb{R}\setminus\mathbb{Z}^-$ is invertible. It was proved that this map is surjective for $\alpha \in \mathbb{C}$. Its injectivity when $\alpha \in \mathbb{R}\setminus\mathbb{Z}^-$ follows from the uniqueness property of the system of Laguerre polynomials with real parameter α not equal to $-1-2, -3, \ldots$ [**(IV.5.3)**]. Notice that if $\alpha > -1$, then the injectivity of $H_{\lambda_0,\alpha}$ follows from the uniqueness property of the Hankel integral transform. Indeed, if $f \equiv 0$, then from (1.8) it follows that $\Phi \sim 0$, i.e. $\Phi(t) = 0$ almost everywhere in $[0,\infty)$ and the identity theorem for holomorphic functions yields that $\Phi \equiv 0$.

There is another proof of the fact that $H_{\lambda_0,\alpha}$ is invertible provided $\alpha > -1$. Define the linear map $H^*_{\lambda_0,\alpha}$ of the space $\mathcal{L}^{(\alpha)}(\lambda_0)$ in the space $G(\lambda_0)$ assuming that to a function $f \in \mathcal{L}^{(\alpha)}(\lambda_0)$ represented in the region $\Delta(\lambda_0)$ by the series [Chapter IV, (2.1)], there corresponds the entire function Φ defined by (1.3).

(VI.1.4) *If* $0 < \lambda_0 \le \infty$ *and* $\alpha > -1$, *then* $\Phi = H^*_{\lambda,\alpha}(f)$ *if and only if*

$$\Phi(w) = \exp w \int_0^\infty t^\alpha \exp(-t) f(t) B_\alpha(wt)\, dt,\ w \in \mathbb{C}. \tag{1.13}$$

In other words, an entire function Φ *is in the space* $G(\lambda_0)$ *if and only if the above representation holds with a function* $f \in \mathcal{L}^{(\alpha)}(\lambda_0)$.

Proof. It is similar to that of the proposition **(VI.1.3)**. Since α is real, the space $\mathcal{L}^{(\alpha)}(\lambda_0)$ coincides with the space $\mathcal{L}(\lambda_0)$ defined by the requirement [Chapter V, (3.9)]. From the last one it follows that if $f \in \mathcal{L}^{(\alpha)}(\lambda_0)$, then $|f(t)| = O(\exp(t/2))$ when t tends to infinity. Then (1.2) yields that the integral on the right-hand side of (1.13) is absolutely uniformly convergent on the compact subsets of the complex plane, hence, Φ is an entire functions.

Since $f \in \mathcal{L}^{(\alpha)}(\lambda_0)$, it is representable in the region $\Delta(\lambda_0)$ by a series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^\infty$. If $\{a_n\}_{n=0}^\infty$ are the coefficients of this series, then from **(IV.2.1)** it follows that the inequality (1.4) holds with $\sigma = \lambda_0$.

Defining $S_\nu(w) = \Phi(w) - \sum_{n=0}^{\nu}(n!)^{-1} a_n w^n$ for $w \in \mathbb{C}$ and $\nu = 0, 1, 2, \ldots$, then [Chapter II, (2.13)] and (1.1) yield that

$$S_\nu(w) = \int_0^\infty t^\alpha \exp(-t) f(t) \left\{ \sum_{n=\nu+1}^\infty \frac{L_n^{(\alpha)}(t)}{\Gamma(n+\alpha+1)} w^n \right\} dt.$$

Further we proceed as in the proof of **(VI.1.3)** and obtain that $\lim_{\nu\to\infty} S_\nu(w) = 0$ for $w \in \mathbb{C}$. It means that the entire function Φ has the representation (1.3), and since the inequality (1.4) holds for its coefficients with $\sigma = \lambda_0$, we can assert that $\Phi \in G(\lambda_0)$.

Conversely, suppose that the function Φ, defined by (1.3), is in the class $G(\lambda_0)$. Then from (1.4) with $\sigma = \lambda_0$ and **(II.2.1)** it follows that the series $\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z)$ is uniformly convergent on the compact subsets of the region $\Delta(\lambda_0)$. Hence, this series defines a complex-valued function $f \in \mathcal{L}^{(\alpha)}(\lambda_0)$ there. But, as we have just seen, in such a case the right-hand side of (1.13) defines an entire function $\Phi^* \in G(\lambda_0)$ and, moreover, $\Phi^*(w) = \sum_{n=0}^{\infty} (n!)^{-1} a_n w^n$, $w \in \mathbb{C}$. Therefore, $\Phi^* \equiv \Phi$ and, hence, Φ has a representation of the kind (1.13) with $f \in \mathcal{L}^{(\alpha)}(\lambda_0)$.

Remark. The fact that the right-hand side of (1.13) has the same form as that of (1.8) is not surprising since a well-known property of the Hankel type transforms is that, in general, they "coincide" with their inverses.

If the entire function Φ defined by (1.3) is in the space $G(\lambda_0)$, then the series [Chapter IV, (2.1)] represents a function $f \in \mathcal{L}^{(\alpha)}(\lambda_0)$ and, moreover, $H^*_{\lambda_0,\alpha}(f) = \Phi$, i.e. the map $H^*_{\lambda_0,\alpha}$ is surjective.

In order to prove the injectivity of this map and that in fact it is the inverse to $H_{\lambda_0,\alpha}$, we have to show that the composition $H_{\lambda_0,\alpha} \circ H^*_{\lambda,\alpha}$ is the identity mapping of the space $\mathcal{L}^{(\alpha)}(\lambda_0)$. If $z \in \Delta(\lambda_0)$, then having in mind [Chapter II, (2.13)], we obtain

$$(1.14) \qquad (H_{\lambda,\alpha} \circ H^*_{\lambda,\alpha})(f)(z) = H_{\lambda,\alpha}(H^*_{\lambda,\alpha}(f))(z) = H_{\lambda,\alpha}(\Phi)(z)$$

$$= \exp z \int_0^\infty t^\alpha \exp(-t) \left\{ \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \right\} B_\alpha(zt)\, dt$$

$$= \exp z \sum_{n=0}^{\infty} \frac{a_n}{n!} \int_0^\infty t^{n+\alpha} \exp(-t) B_\alpha(zt)\, dt = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z) = f(z).$$

2. Meijer transform and the representation by series in Laguerre associated functions

2.1 The equality [Chapter II, (4.7)] shows that the function $-(-z)^{-\alpha/2} M_n^{(\alpha)}(z)$ which is holomorphic in the region $\mathbb{C} \setminus [0,\infty)$, is a Meijer's type integral transform of the function $2(\Gamma(n+1))^{-1} z^{n+\alpha/2} \exp(-z)$. Therefore, one may expect that if a complex-valued function is representable by a series in Laguerre associated functions, then it is a Meijer transform of a suitable (holomorphic) function. This is, really, the fact and, more precisely, we have the following proposition:

(VI.2.1) *Suppose that* $0 \le \mu_0 < \infty$ *and* $\alpha > -1$. *A complex-valued function* f *is in the class* $\mathcal{M}^{(\alpha)}(\mu_0)$ *if and only if for it an integral representation of the kind*

$$f(z) = -2(-z)^{\alpha/2}\int_0^\infty t^{\alpha/2}\exp(-t)\Psi(t)K_\alpha(2\sqrt{-zt})\,dt,\ z \in \Delta^*(\mu_0) \tag{2.1}$$

holds, where K_α *is the modified Bessel function of the third kind with index* α [Appendix, (2.6)] *and the entire function* $\Psi \in G(\mu_0)$.

Proof. Notice that if $\Psi \in G(-\mu_0), 0 \le \mu_0 < \infty$, then the integral in the right-hand side of (2.1) is absolutely uniformly convergent on every compact set $E \subset \Delta^*(\mu_0)$. Indeed, let $\mu \in (\mu_0, \infty)$ be chosen so that $E \subset \Delta^*(\mu)$ and denote $\delta = (\mu - \mu_0)/2$. Then the estimate $|\Psi(t)| = O\{\exp(t + 2(\mu_0 + \delta)\sqrt{t})\},\ t \to \infty$, as well as the asymptotic formula [Appendix, (2.17)], yield that

$$t^{\alpha/2}\exp(-t)|\Psi(t)K_\alpha(2\sqrt{-zt})| = O\{t^{\alpha/2}\exp(-2\delta t)\}$$

for $z \in E$.

Suppose that the representation (2.1) holds for the function f with $\Psi \in G(-\mu_0)$. If Ψ has the expansion

$$\Psi(w) = \sum_{n=0}^{\infty}(n!)^{-1}b_n w^n, \tag{2.2}$$

then **(VI.1.2)** and **(II.2.3), (b)** yield that the series in the Laguerre associated functions $\{M_n^{(\alpha)}(z)\}_{n=0}^\infty$ with coefficients $\{b_n\}_{n=0}^\infty$ is convergent in the region $\Delta^*(\mu_0)$.

Define $\rho_\nu(z) = f(z) - \sum_{n=0}^{\nu} b_n M_n^{(\alpha)}(z)$ for $z \in \Delta^*(\mu_0)$. Then the integral representation [Chapter II, (4.7)] yields

$$\rho_\nu(z) = -2(-z)^{\alpha/2}\int_0^\infty t^{\alpha/2}\exp(-t)\left\{\sum_{n=\nu+1}^{\infty}(n!)^{-1}b_n t^n\right\}K_\alpha(2\sqrt{-zt})\,dt$$

for $z \in \Delta^*(\mu_0)$. Further, we proceed as in the proof of **(VI.1.4)**, i.e. in view of the fact that the entire function $\Psi^*(w) = \sum_{n=0}^{\infty}(n!)^{-1}|b_n|w^n$ is in the class $G(-\mu_0)$, we conclude that if $\varepsilon > 0$, then $\rho_\nu(z) = O(\varepsilon)$ provided that $\nu \ge N$ and that the positive integer $N = N(\varepsilon)$ is large enough. Hence, the series

$$\sum_{n=0}^{\infty} b_n M_n^{(\alpha)}(z) \tag{2.3}$$

represents the function f in the region $\Delta^*(\mu_0)$, i.e. $f \in \mathcal{M}^{(\alpha)}(\mu_0)$.

If, conversely, the function $f \in \mathcal{M}^{(\alpha)}(\mu_0), 0 \leq \mu_0 < \infty$, is represented by the series (2.3), then **(IV.2.3)** and **(VI.1.2)** yield that the entire function Ψ defined by (2.2) is in the class $G(-\mu_0)$. The right-hand side of (2.1) defines a complex function $\tilde{f} \in \mathcal{H}(\Delta^*(\mu_0))$ and, moreover, for $z \in \Delta^*(\mu_0)$ the function $\tilde{f}$ is representable by the series (2.3), i.e. $\tilde{f} \equiv f$.

2.3 Suppose that $D \subset \mathbb{C} \setminus \{0\}$ is an unbounded region and f is a complex-valued function defined in D. Suppose that there exists a sequence $\{A_n\}_{n=0}^{\infty}$ of complex numbers such that $\lim_{z\in D, z\to\infty} z^{\nu} R_{\nu}(z) = 0$ for $\nu = 0, 1, 2, \ldots$, where $R_{\nu}(z) = f(z) - \sum_{n=0}^{\nu} A_n z^{-n}$, $\nu = 0, 1, 2, \ldots$. Then, by definition, the series

$$\sum_{n=0}^{\infty} \frac{A_n}{z^n} \tag{2.4}$$

is called asymptotic expansion of the function f when $z \to \infty$ in the region D. In such a case we write

$$f(z) \sim \sum_{n=0}^{\infty} \frac{A_n}{z^n}, \ z \to \infty. \tag{2.5}$$

We point out that the series (2.4) does not need to be convergent in D or even in the vicinity of the point of infinity.

An asymptotic expansion, if it exists, is unique. This follows immediately from the equalities:

$$A_0 = \lim_{z\in D, z\to\infty} f(z), \ A_n = \lim_{z\in D, z\to\infty} z^n \left\{ f(z) - \sum_{k=0}^{n-1} \frac{A_k}{z^k} \right\}, \ n = 1, 2, 3, \ldots. \tag{2.6}$$

A series of the kind (2.4) may well be an asymptotic expansion of more than one function. There is a criterion which enable us to assert that (2.4) is an asymptotic expansion of exactly one complex-valued function. It is based on the following proposition:

(VI.2.2) *Suppose that $\rho > 0, 0 \leq \theta < 2\pi, 0 < \delta < 1$ and let F be a complex-valued function which is holomorphic in the region $S(\rho, \theta, \delta) = \{z \in \mathbb{C} : |z| > \rho, |\arg[\exp(-i\theta)z]| < \delta\pi\}$.*

Suppose that there exist two increasing sequences of positive numbers $\{\mu_n\}_{n=0}^{\infty}$ and $\{\nu_n\}_{n=0}^{\infty}$ with $\lim_{n\to\infty} \nu_n = \infty$, a positive integer n_0, and a nonnegative constant M such that the inequality $|F(z)| \leq M(\mu_n |z|^{-1/2\delta})^{\nu_n}$ holds for $z \in S(\rho, \theta, \delta)$ and $n \geq n_0$.

If the series $\sum_{n=0}^{\infty} (\nu_{n+1} - \nu_n)\mu_n^{-1}$ diverges, then F is identically zero in $S(\rho, \theta, \delta)$.

As a corollary we have the proposition:

(VI.2.3) *Let the complex-valued function f be holomorphic in the region $S(\rho,\theta,\delta)$ and let (2.4) be its asymptotic expansion there. If $|R_n(z)| = O((\mu_n|z|^{-1/2\delta})^{\nu_n})$ uniformly for $z \in S(\rho,\theta,\delta)$ and $n \geq n_0$, then f is the unique holomorphic function having (2.4) as its asymptotic expansion in $S(\rho,\theta,\delta)$. In particular, this is true if $|R_n(z)| = O(\Gamma(\lambda_n+1)(\sigma|z|^{-1/2\delta})^{\lambda_n}, \sigma > 0$, uniformly for $z \in S(\rho,\theta,\delta)$ and $n \geq n_0$ provided $\{\lambda_n\}_{n=1}^{\infty}$ is an increasing sequence of positive numbers such that $\lim_{n\to\infty} \lambda_n = \infty$.*

Using the asymptotic expansion's "language" it is possible to state a proposition which gives a necessary as well as sufficient condition for a complex-valued function to be in a space of the kind $\mathcal{M}^{(\alpha)}(\mu_0)$.

(VI.2.4) *If $0 \leq \mu_0 < \infty, -1/2 \leq \alpha \leq 1/2$, and $f \in \mathcal{M}^{(\alpha)}(\mu_0)$, then for every $\mu > \mu_0$ the function f has in $\Delta^*(\mu)$ the asymptotic expansion*

$$f(z) \sim \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+\alpha+1)c_n}{z^{n+1}},\ z \to \infty, \tag{2.7}$$

where

$$\limsup_{n\to\infty} (2\sqrt{n})^{-1} \log\left|\sum_{k=0}^{n} \binom{n}{k} c_k\right| \leq \mu_0. \tag{2.8}$$

Conversely, if $f \in \mathcal{H}(\Delta^*(\mu_0))$ and the equality

$$\lim_{z\in S(\rho,\pi,1/2), z\to\infty} z^n \left\{ f(z) - \sum_{k=0}^{n-1} \frac{(-1)^k \Gamma(\alpha+k+1)c_k}{z^{k+1}} \right\} = 0 \tag{2.9}$$

holds uniformly with respect to $n \geq n_0$ for some $\rho > \mu_0^2$, then $f \in \mathcal{M}^{(\alpha)}(\mu_0)$.

Proof. Suppose that $-1/2 < \alpha < 1/2$ and let the function $f \in \mathcal{M}^{(\alpha)}(\mu_0)$, $0 \leq \mu_0 < \infty$, have the expansion [Chapter IV, (5.17)] in the region $\Delta^*(\mu_0)$. Then the representation (2.1) holds, where the entire function $\Psi \in G(-\mu_0)$ is given by the series (2.2).

If we define $a(z) = z\Psi(z^2)\exp(-z^2)$, then from (1.1) it follows that $|a(t)| = O(\exp(2\mu t))$ for $\mu > \mu_0$ when t tends to infinity. Hence, the integral

$$A(w) = \int_0^{\infty} a(t)\exp(-wt)\,dt \tag{2.10}$$

is uniformly convergent on each closed half-plane $\operatorname{Re} w \geq 2\mu$ with $\mu > \mu_0$, i.e. it defines a holomorphic function in the half-plane $\operatorname{Re} w > 2\mu_0$.

From **(VI.1.2)** we obtain that the function $\Psi^{(k)}$ is in the class $G(-\mu_0)$ for $k = 1, 2, 3, \ldots$. Further, it is easy to prove that if t tends to infinity, then

$$|a^{(k)}(t)| = O(\exp(2\mu t)) \tag{2.11}$$

for $k = 1, 2, 3, \ldots$ and $\mu \geq \mu_0$. In particular, since $a(0) = 0$, from (2.10) it follows that for $\mu > \mu_0$

$$|A(w)| = O(|w|^{-1}) \tag{2.12}$$

when w tends to infinity in the half-plane $\operatorname{Re} w \geq 2\mu$.

Using the integral representations (2.1) and [Appendix, (2.16)], and replacing z by $-z^2/4$, and t by t^2, we find that if $\operatorname{Re} z > 2\mu_0$, then

$$f(-z^2/4) = -\frac{4\sqrt{\pi}}{\Gamma(\alpha + 1/2)} \int_0^\infty a(t)\, dt \int_1^\infty (\tau^2 - 1)^{\alpha - 1/2} \exp(-zt\tau)\, d\tau.$$

Having in mind (2.12), as well as the assumption $\alpha < 1/2$, we easily prove that the multiple integral

$$\int_1^\infty (\tau^2 - 1)^{\alpha - 1/2}\, d\tau \int_0^\infty a(t) \exp(-zt\tau)\, dt$$

is absolutely convergent in the half-plane $\operatorname{Re} z > 2\mu_0$. Then, changing the order of integrations, we come to the representation

$$f(-z^2/4) = -\frac{4\sqrt{\pi}}{\Gamma(\alpha + 1/2)} \int_1^\infty (\tau^2 - 1)^{\alpha - 1/2} A(z\tau)\, d\tau, \ \operatorname{Re} z > 2\mu_0. \tag{2.13}$$

Since (2.11) holds for each $\mu > \mu_0$, the integration by parts in (2.10) is possible. Thus, we find that if $\operatorname{Re} w > 2\mu$, then

$$A(w) = \sum_{k=0}^{2n} \frac{a^{(k)}(0)}{w^{k+1}} + \frac{1}{w^{2n+1}} \int_0^\infty a^{(2n+1)}(t) \exp(-wt)\, dt \tag{2.14}$$

Further, we have the expansion $a(z) = \sum_{n=0}^{\infty} c_n (n!)^{-1} z^{2n+1}$, where

$$c_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} b_{n-k}, \ n = 0, 1, 2, \ldots. \tag{2.15}$$

Hence,

$$A(w) = \sum_{k=0}^{n-1} \frac{(2k+1)!c_k}{k!w^{2k+2}} + \frac{1}{w^{2n+1}} \int_0^\infty a^{(2n+1)}(t) \exp(-wt)\, dt$$

and then (2.13) yields that

$$f(-z^2/4) = \sum_{k=0}^{n-1} \frac{2^{2k+2}\Gamma(k+\alpha+1)c_k}{z^{2k+2}} + \tilde{R}_n(z),\ \operatorname{Re} z > 2\mu,$$

where

$$\tilde{R}_n(z) = \frac{4\sqrt{\pi}}{\Gamma(\alpha+1/2)z^{2n+1}} \int_1^\infty (\tau^2-1)^{\alpha-1/2}\tau^{-2n-1}\,d\tau \int_0^\infty a^{(2n+1)}(t)\exp(-z\tau t)\,dt.$$

In this way we obtain the representation

$$f(z) = \sum_{k=0}^{n-1} \frac{(-1)^k\Gamma(k+\alpha+1)c_k}{z^{k+1}} + R_n(z),\ z \in \Delta^*(\mu),\ n = 1,2,3,\dots,$$

where $R_n(z) = \tilde{R}_n(2(-z)^{1/2}),\ z \in \Delta^*(\mu)$. Since $\lim_{z\in\Delta^*(\mu),z\to\infty} z^n R_n(z) = 0$, it follows that the asymptotic expansion (2.7) holds whenever $\mu > \mu_0$.

From (2.15) it follows

$$\sum_{k=0}^{n} \binom{n}{k} c_k = b_n,\ n = 0,1,2,\dots, \tag{2.16}$$

and since the series in the right-hand side of [Chapter IV, (5.17)] converges in the region $\Delta^*(\mu_0)$, **(IV.2.3),(b)** implies (2.10).

Suppose that $\alpha = 1/2$. Since $K_{1/2}(z) = \sqrt{\pi/2z}\exp(-z)$ [Appendix, (2.13)], we have

$$f(-z^2/4) = -2\sqrt{\pi}\int_0^\infty t\exp(-t^2)\Psi(t^2)\exp(zt)\,dt = -2\sqrt{\pi}A(z).$$

Then (2.14) gives

$$f(-z^2/4) = -2\sqrt{\pi}\sum_{k=0}^{n-1}\frac{(2k+1)!c_k}{k!z^{2k+2}} - \frac{2\sqrt{\pi}}{z^{2n+1}}\int_0^\infty a^{(2n+1)}(t)\exp(-zt)\,dt,$$

i.e. the representation

$$f(z) = \sum_{k=0}^{n-1}\frac{(-1)^k\Gamma(k+3/2)c_k}{z^{k+1}} + o(|z|^{-n}),\ z \in \Delta^*(\mu),\ z \to \infty$$

holds whenever $\mu > \mu_0$. The proof in the case $\alpha = -1/2$ is similar to that when $\alpha = 1/2$.

Suppose that (2.9) holds uniformly for $n \ge n_0$ and, moreover, that the inequality (2.8) is true. Then we have

$$\lim_{z\in S(\rho,\pi,1/2),z\to\infty} z^{n+1}R_n^*(z) = (-1)^n\Gamma(n+\alpha+1)c_n,\ n=1,2,3,\dots \tag{2.17}$$

uniformly for $n \ge n_0$, where

$$R_n^*(z) = f(z) - \sum_{k=0}^{n-1} \frac{(-1)^k\Gamma(k+\alpha+1)c_k}{z^{k+1}},\ n=1,2,3,\dots.$$

From (2.16) and (2.8) we obtain

$$\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|b_n| \le \mu_0, \tag{2.18}$$

i.e. $|b_n| = O(\exp(2\mu\sqrt{n}))$ for $\mu > \mu_0$ when n tends to infinity and, hence, the relation (2.15) implies $|c_n| = O(2^n\exp(2\mu\sqrt{n}))$ when n tends to infinity. Then, having in mind (2.17), we find

$$|R_n^*(z)| = O\{\Gamma(n+\alpha+1)(2|z|^{-1})^{n+\alpha}\},\ z\in S(\rho,\pi,1/2),\ z\to\infty,$$

uniformly for $n \ge n_0$. Further, from **(VI.2.3)** it follows that f is the only holomorphic function having the right-hand side of (2.7) as its asymptotic expansion in the region $S(\rho,\pi,1/2)$.

From (2.18) and **(IV.2.3), (b)** we obtain that the series in the Laguerre associated functions $\{M_n^{(\alpha)}(z)\}_{n=0}^\infty$, $-1/2 \le \alpha \le 1/2$, with coefficients $\{b_n\}_{n=0}^\infty$ converges in the region $\Delta^*(\mu_0)$ and, hence, its sum $\tilde{f}$ is a holomorphic function there. Since $\tilde{f}$ has the same asymptotic expansion in $S(\rho,\pi,1/2)$ as the function f, it follows that $f \equiv \tilde{f}$, i.e. f is representable in the region $\Delta^*(\mu_0)$ as a series in the Laguerre associated functions $\{M_n^{(\alpha)}(z)\}_{n=0}^\infty$.

3. Laplace transform and the representation by series in Laguerre associated functions

3.1 There is another way to make use of a classical integral transform when the representation problem by series in Laguerre associated functions is in question. Indeed, the representation [Chapter IV, (4.1)] shows that if $\operatorname{Re} z < 0$, then the function $-(-z)^{-\alpha}M_n^{(\alpha)}(z)$ is a Laplace transform of the function $z^{n+\alpha}(1+z)^{-n-1}$ which is holomorphic in the region $\mathbb{C}\setminus(-\infty,0]$. This observation leads to the conjecture that if a holomorphic function is representable by a series in Laguerre associated functions, then it is a Laplace transform of a suitable (holomorphic) function.

In order to clarify the above considerations we introduce the class $\mathcal{B}(\sigma)$,

$0 \le \sigma < \infty$, of all complex-valued functions $B(z)$ which are holomorphic in the half-plane $V(1/\sqrt{2}) = \{w \in \mathbb{C} : \operatorname{Re} w > -1/2\}$ and have the property that for each $\varepsilon > 0$ there exists a positive constant $b(\varepsilon)$ such that

$$|B(w)| \le b(\varepsilon) \exp\left\{ \frac{(\sigma+\varepsilon)^2 |w|}{|1+w| - |w|} \right\}, \; w \in V. \tag{3.1}$$

(VI.3.1) *A function*

$$B(w) = \sum_{n=0}^{\infty} b_n \Big(\frac{w}{1+w} \Big) \in \mathcal{H}(V(1/\sqrt{2}))$$

is in the class $\mathcal{B}(\sigma)$ if and only if

$$\limsup_{n\to\infty} (2\sqrt{n})^{-1} \log |b_n| \le \sigma. \tag{3.2}$$

Proof. By the mapping $\zeta = w(1+w)^{-1}$ the half-plane $V(1/\sqrt{2})$ is transformed into the unit disk $U(0;1)$ and to a function $B \in \mathcal{H}(V(1/\sqrt{2}))$ there corresponds the function $\tilde{B}(\zeta) = B(\zeta(1-\zeta)^{-1}) \in \mathcal{H}(U(0;1))$. In particular, the class $\mathcal{B}(\sigma)$ is transformed into the class $\tilde{\mathcal{B}}(\sigma)$ of functions $\tilde{B} \in \mathcal{H}(U(0;1))$ with the property that for each $\varepsilon > 0$ there exists a positive constant $\tilde{b}(\varepsilon)$ such that

$$|\tilde{B}(\zeta)| \le \tilde{b}(\varepsilon) \exp\left\{ \frac{(\sigma+\varepsilon)^2 |\zeta|}{1-|\zeta|} \right\} \tag{3.3}$$

Therefore, we have to prove that a function $\tilde{B}(\zeta) = \sum_{n=0}^{\infty} b_n \zeta^n \in \mathcal{H}(U(0;1))$ is in the class $\tilde{\mathcal{B}}(\sigma)$ if and only if the inequality (3.2) holds.

The necessity of (3.2) follows from the Cauchy inequalities for the coefficients of the power series expansion of a holomorphic function. Indeed, if $\varepsilon > 0$ and $r_n = 1 - (\sigma+\varepsilon) n^{-1/2}, n > (\sigma+\varepsilon)^2$, then $r_n^{-n} = O\{\exp((\sigma+\varepsilon)\sqrt{n})\}$ when n tends to infinity. Hence,

$$|b_n| = O\left\{ \int_{|\zeta|=r_n} |\tilde{B}(\zeta)\zeta^{-n-1}|\, ds \right\} = O\left\{ r_n^{-n} \exp \frac{(\sigma+\epsilon)^2 r_n}{1-r_n} \right\}$$

$$= O\{\exp(2(\sigma+\epsilon)\sqrt{n})\}.$$

The sufficiency can be proved as an application of the expansion [Chapter II, (2.4)] and the asymptotic properties of the Laguerre polynomials.

We define the function $\tilde{B}_\delta(\zeta) = \exp(\delta^2 \zeta (1-\zeta)^{-1})$ for $\delta > 0$ and $\zeta \in U(0;1)$. Then [II, (2.4)] yields that $\tilde{B}_\delta(\zeta) = \sum_{n=0}^{\infty} L_n^{(-1)}(-\delta^2)\zeta^n, \; \zeta \in U(0;1)$.

Suppose that the sequence $\{b_n\}_{n=0}^{\infty}$ satisfies (3.2). Using the asymptotic formula [Chapter III, (2.6)] for the Laguerre polynomials with parameter $\alpha = -1$ [**(III.2.3)**], we find that

$$\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log\{|b_n||L_n^{(-1)}(-(\sigma+\varepsilon)^2)|^{-1}\}$$

$$= \limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|b_n| - \lim_{n\to\infty}(2\sqrt{n})^{-1}\log|L_n^{(-1)}(-(\sigma+\varepsilon)^2)| \le \sigma-(\sigma+\varepsilon) = -\varepsilon,$$

i.e. $|b_n| = O\{L_n^{(-1)}(-(\sigma+\epsilon)^2)\}$ for each positive ε. Since the Taylor coefficients of $\tilde{B}_\delta(\zeta)$ are positive, we obtain

$$|\tilde{B}(\zeta)| \le \sum_{n=0}^{\infty}|b_n||\zeta|^n = O\Big\{\sum_{n=0}^{\infty}L_n^{(-1)}(-(\sigma+\varepsilon)^2)|\zeta|^n\Big\}$$

$$= O(\tilde{B}_{\sigma+\varepsilon}(|\zeta|)) = O\Big\{\exp\frac{(\sigma+\varepsilon)^2|\zeta|}{1-|\zeta|}\Big\},\ \zeta\in U(0;1),$$

i.e. the function $\tilde{B} \in \tilde{\mathcal{B}}(\sigma)$.

For a complex-valued function F which is holomorphic in the region $V(1/\sqrt{2})\setminus(-1/2,0])$, i.e. in the half-plane $\operatorname{Re} w > 1/2$ cut along the segment $(-1/2,0]$, we say that it is in the class $\mathcal{B}(\sigma,\alpha), 0\le\sigma<\infty, \alpha>-1$, if the function $w^{-\alpha}F(w)$ is in the class $\mathcal{B}(\sigma)$. It is easy to see that $F\in\mathcal{B}(\sigma,\alpha)$ if and only if $F(w) = (1+w)^{-1}w^\alpha B(w)$, where $B\in\mathcal{B}(\sigma)$. Using the last representation of the functions in $\mathcal{B}(\sigma,\alpha)$, as well as **(VI.3.1)**, we shall prove the following proposition:

(VI.3.2) *Suppose that $0\le\mu_0<\infty$ and $\alpha>-1$. A complex-valued function $f\in\mathcal{H}(\Delta^*(\mu_0))$ has an expansion in a series of the Laguerre associated functions $\{M_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ in the region $\Delta(\mu_0)$ if and only if the representation*

$$f(z) = -(-z)^\alpha\int_0^\infty F(t)\exp(zt)\,dt \tag{3.4}$$

holds in the halfplane $\operatorname{Re} z < -\mu_0^2$ with a function $F\in\mathcal{B}(\mu_0,\alpha)$.

Proof. If $f(z) = \sum_{n=0}^{\infty}b_nM_n^{(\alpha)}(z)$ for $z\in\Delta^*(\mu_0)$, then from **(IV.2.3)** it follows that the sequence $\{b_n\}_{n=0}^{\infty}$ satisfies (3.2) with $\sigma=\mu_0$. Then, from **(VI.3.1)**, we obtain that the function

$$F(w) = (1+w)^{-1}w^\alpha\sum_{n=0}^{\infty}b_n\Big(\frac{w}{1+w}\Big)^n \tag{3.5}$$

as well as the function

$$F^*(w) = (1+w)^{-1}w^\alpha\sum_{n=0}^{\infty}|b_n|\Big(\frac{w}{1+w}\Big)^n$$

is in the class $\mathcal{B}(\mu_0, \alpha)$. In particular, if $\tau > 0$, then $F(t) = O\{\exp((\mu_0+\tau)^2 t)\}$ and $F^*(t) = O\{\exp((\mu_0+\tau)^2 t)\}$ when t tends to infinity. Hence, the integral in (3.4) is absolutely uniformly convergent on each (closed) half-plane $\operatorname{Re} z \le -\mu^2$ with $\mu \in (\mu_0, \infty)$, i.e. this integral defines a complex-valued function holomorphic in the half-plane $\operatorname{Re} z < -\mu_0^2$.

We define

$$\eta_\nu(z) = -(-z)^\alpha \int_0^\infty F(t)\exp(zt)\,dt - \sum_{n=0}^{\nu} b_n M_n^{(\alpha)}(z),\ \operatorname{Re} z < -\mu_0,\ \nu = 0,1,2,\dots.$$

Then, having in mind (3.5) and [Chapter II, (4.1)], we find that

$$\eta_\nu(z) = -(-z)^\alpha \int_0^\infty \frac{t^\alpha}{1+t}\Big\{\sum_{n=\nu+1}^{\infty} b_n\Big(\frac{t}{1+t}\Big)^n\Big\}\exp(zt)\,dt. \tag{3.6}$$

If $x = Rez < -\mu_0^2$, then we choose $\tau > 0$ such that $(\mu_0+\tau)^2 + x < 0$. Further, for every $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that

$$\int_T^\infty \exp\{((\mu_0+\tau)^2 + x)t\}\,dt < \varepsilon.$$

Then we obtain

$$\Big|\int_T^\infty \frac{t^\alpha}{1+t}\Big\{\sum_{\nu+1}^{\infty} b_n\Big(\frac{t}{1+t}\Big)^n\Big\}\exp(zt)\,dt\Big| \tag{3.7}$$

$$\le \int_T^\infty \frac{t^\alpha}{1+t}\Big\{\sum_{\nu+1}^{\infty} |b_n|\Big(\frac{t}{1+t}\Big)^n\Big\}\exp(xt)\,dt$$

$$\le \int_T^\infty F^*(t)\exp(xt) = O\Big(\int_T^\infty \exp\{((\mu_0+\tau)^2+x)t\}\,dt\Big) = O(\epsilon)$$

for $\nu = 0,1,2,\dots$.

Further, we choose $N = N(\varepsilon) > 0$ such that $\Big|\sum_{n=\nu+1}^{\infty} b_n(t(1+t)^{-1})^n\Big| < \varepsilon$ for $\nu > N$ and $0 \le t \le T$. Then $(\nu > N)$

$$\Big|\int_0^T \frac{t^\alpha}{1+t}\Big\{\sum_{n=\nu+1}^{\infty} b_n\Big(\frac{t}{1+t}\Big)^n\Big\}\exp(zt)\,dt\Big| \tag{3.8}$$

$$= O\Big(\varepsilon\int_0^T \frac{t^\alpha \exp(xt)}{1+t}\,dt\Big) = O(\varepsilon).$$

From (3.6), (3.7) and (3.8) it follows that $\eta_\nu(z) = O(\varepsilon)$, $\nu > N$, and the validity of the representation (3.4) is proved.

Suppose that the complex-valued function $f \in \mathcal{H}(\Delta^*(\mu_0))$ has the representation (3.4) in the half-plane $\operatorname{Re} z < -\mu_0^2$ with $F \in \mathcal{B}(\mu_0, \alpha)$. If F is defined by the expansion (3.5), then from **(VI.3.1)** it follows that (3.2) holds with $\sigma = \mu_0$. By virtue of **(IV.2.3)**, the series in the Laguerre associated functions $\{M_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ with the same coefficients as those of expansion (3.5) converges to a holomorphic function $\tilde{f}$ in the region $\Delta^*(\mu_0)$. But, as it was just proved, $\tilde{f}$ has a representation of the kind (3.4) in the half-plane $\operatorname{Re} z < -\mu_0^2$. Hence, $\tilde{f} \equiv f$, i.e f has an expansion in a series of Laguerre associated functions with parameter α in the region $\Delta^*(\mu_0)$.

Examples: **(1)** If $0 < \mu_0 < \infty$ and $\alpha > -1$, then $F(w) = w^{\alpha} \exp(\mu_0 w)$ $\in \mathcal{B}(\mu_0, \alpha)$. Moreover, from [Chapter II, (2.2)] it follows that

$$F(w) = (1+w)^{-1} w^{\alpha} \sum_{n=0}^{\infty} L_n^{(0)}(-\mu_0^2) \Big(\frac{t}{1+t}\Big)^n.$$

Since

$$\exp(2\alpha\pi i)\Gamma(\alpha+1)(z+\mu_0^2)^{-1}\Big(\frac{z}{z+\mu_0^2}\Big)^{\alpha} = -(-z)^{\alpha} \int_0^{\infty} t^{\alpha} \exp((z+\mu_0^2)t)\, dt$$

in the half-plane $\operatorname{Re} z < -\mu_0^2$, from **(VI.3.2)** it follows that if $z \in \Delta^*(\mu_0)$, then

$$\sum_{n=0}^{\infty} L_n^{(0)}(-\mu_0^2) M_n^{(\alpha)}(z) = \exp(2\alpha\pi i)\Gamma(\alpha+1)(z+\mu_0^2)^{-1}\Big(\frac{z}{z+\mu_0^2}\Big)^{\alpha}. \tag{3.9}$$

(2) Suppose that the function $F(t), 0 \le t < \infty$ satisfies the conditions of proposition **(V.4.2)** with $\alpha = 0$ and define

$$\tilde{F}(w) = \frac{1}{2\pi i} \int_0^{\infty} F(t) \exp(-wt)\, dt. \tag{3.10}$$

Since $F(t) = O(t^{-1/2} \exp(-t/2))$ for $t \to \infty$, $\tilde{F}$ is a holomorphic function in the half-plane $\operatorname{Re} w > -1/2$ and, moreover, $\tilde{F} \in \mathcal{B}(\varepsilon, 0)$ for each positive ε. If f is the function, defined by [Chapter V, (4.2)], and $\operatorname{Re} z < -\varepsilon^2$, then (3.10) yields that

$$-\int_0^{\infty} \tilde{F}(u) \exp(zu)\, du = -\frac{1}{2\pi i} \int_0^{\infty} F(t) \Big\{ \int_0^{\infty} \exp((z-t)u)\, du \Big\}\, dt$$

$$= -\frac{1}{2\pi i} \int_0^{\infty} \frac{F(t)}{t-z}\, dt = f(z),$$

i.e. if $\alpha = 0$, then the proposition **(V.4.2)** follows from **(VI.3.2)**.

3.2 If $0 < \lambda \le \infty$ and $\alpha \in \mathbb{R}$, then a function $f \in \mathcal{H}(\Delta(\lambda_0))$ has a representation for $z \in \Delta(\lambda_0)$ by a series in Laguerre polynomials with parameter α if and only if it is in the vector space $\mathcal{L}(\lambda_0)$ [**(V.3.7)**]. But if the functions f and g are in $\mathcal{L}(\lambda_0)$, then, in general, it is not true that $fg \in \mathcal{L}(\lambda_0)$. For instance, if $a, b < 1/2$ and $a + b = 1/2$, then the entire functions $\exp(az)$ and $\exp(bz)$ are in $\mathcal{L}(\infty)$ but $\exp(az)\exp(bz) = \exp(z/2)$ is not in $\mathcal{L}(\infty)$.

In the case of Laguerre associated functions the situation is quite different. Before stating a general result, we need an auxiliary proposition:

(VI.3.3) *If $F \in \mathcal{B}(\sigma, \alpha), G \in \mathcal{B}(\tau, \beta),\ 0 \le \sigma, \tau < \infty,\ \alpha, \beta > -1$, then the function*

$$H(w) = -\int_0^w F(\zeta)G(w-\zeta)\,d\zeta \tag{3.11}$$

is in $\mathcal{B}(\max(\sigma, \tau), \alpha + \beta + 1)$.

Proof. It is easy to see that if $\operatorname{Re} w < -1/2$ and $0 \le t \le 1$, then

$$(|1 + wt| - |wt|)^{-1} \le (|1 + w| - |w|)^{-1}. \tag{3.12}$$

Indeed, if $\operatorname{Re} w > -1/2$, then the function $\varphi(t) = |1+wt|-|wt|$ is positive and decreasing on the segment $[0, 1]$. Hence, for each $t \in [0, 1]$ we have the inequality $\varphi(t) \ge \varphi(1)$ which is equivalent to (3.12).

Further, if $F(w) = w^\alpha B(w)$ and $G(w) = w^\beta C(w)$, with $B \in \mathcal{B}(\sigma)$ and $C \in \mathcal{B}(\tau)$, then from the definition of the function $H(w)$ by (3.11) it follows that $H(w) = -w^{\alpha+\beta+1}D(w)$, where

$$D(w) = \int_0^1 t^\alpha (1-t)^\beta B(wt)C(w(1-t))\,dt.$$

If $\varepsilon > 0$, then there exist $b(\varepsilon) > 0$ and $c(\varepsilon) > 0$ such that the inequality (3.1) as well as the inequality

$$|C(w)| \le c(\varepsilon)\exp\left\{\frac{(\tau+\varepsilon)^2|w|}{|1+w|-|w|}\right\}$$

holds. Then, having in mind(3.12), we obtain that

$$|D(w)| \le d(\varepsilon)\exp\left\{\frac{(\max(\sigma,\tau)+\varepsilon)^2|w|}{|1+w|-|w|}\right\},$$

where

$$d(\varepsilon) = b(\varepsilon)c(\varepsilon)\int_0^1 t^\alpha(1-t)^\beta\,dt.$$

(VI.3.4) *Suppose that $0 \le \sigma, \tau < \infty,\ \alpha, \beta > -1$ and $\alpha + \beta > -1$. If $f \in \mathcal{M}^{(\alpha)}(\sigma)$ and $g \in \mathcal{M}^{(\beta)}(\tau)$, then $fg \in \mathcal{M}^{(\alpha+\beta)}(\max(\sigma, \tau))$.*

Proof. From **(VI.3.2)** it follows that

$$f(z) = -(-z)^{\alpha} \int_0^{\infty} F(t) \exp(zt)\, dt, \;\; \operatorname{Re} z < -\sigma^2,$$

$$g(z) = -(-z)^{\beta} \int_0^{\infty} G(t) \exp(zt)\, dt, \;\; \operatorname{Re} z < -\tau^2,$$

where $F \in \mathcal{B}(\sigma, \alpha)$ and $G \in \mathcal{B}(\tau, \beta)$. The multiplication rule for the Laplace transform yields that

$$f(z)g(z) = -(-z)^{\alpha+\beta} \int_0^{\infty} H(t) \exp(zt)\, dt, \;\; \operatorname{Re} z < -(\max(\sigma, \tau))^2,$$

where H is defined by (3.11). Moreover, from **(VI.3.3)** we obtain that $H \in \mathcal{B}(\max(\sigma, \tau), \alpha+\beta+1) \subset \mathcal{B}(\max(\sigma, \tau), \alpha+\beta)$. Therefore, by virtue of **(VI.3.2)**, the function fg has a representation as a series in the Laguerre associated functions $\{M_n^{(\alpha+\beta)}(z)\}_{n=0}^{\infty}$ in the region $\Delta^*(\max(\sigma, \tau))$.

Corollary. *If $f, g \in \mathcal{M}^{(0)}(\mu_0), 0 \leq \mu_0 < \infty$, then $fg \in \mathcal{M}^{(0)}(\mu_0)$, i.e. every space $\mathcal{M}^{(0)}(\mu_0), 0 \leq \mu_0 < \infty$, is an algebra over the field of complex numbers.*

4. Fourier transform and the representation by series in Hermite polynomials and associated functions

4.1. As in the Laguerre case there is another approach to the representation problem by series in the Hermite polynomials. In a similar way, i.e. using the integral representation [Chapter II, (3.4)] or [Chapter II, (3.5)] of Hermite polynomials, we can prove a proposition like **(VI.1.4)**. But it is much more convenient to make use of the last proposition as well as of the relations between Hermite and Laguerre polynomials [Chapter I, (5.24), (5.25)]. Recall that by $\mathcal{E}$ we denote the $\mathbb{C}$-vector space of complex-valued functions which are holomorphic in the strip $S(\tau_0) = \{z \in \mathbb{C} : |\operatorname{Im} z| < \tau_0\}, \; 0 < \tau_0 \leq \infty$, and have there representations by series in Hermite polynomials.

(VI.4.1) *Suppose that $0 < \tau_0 \leq \infty$. A complex-valued function f is in the space $\mathcal{E}(\tau_0)$ if and only if the representation*

$$(4.1) \qquad f(z) = \int_{-\infty}^{\infty} \{\Phi_1(t^2) + t\Phi_2(t^2)\} \exp(-(t - iz)^2)\, dt$$

$$= 2 \exp z^2 \int_0^{\infty} \exp(-t^2)\{\Phi_1(t^2) \cos(2zt) + it\Phi_2(t^2) \sin(2zt)\}\, dt$$

holds in the strip $S(\tau_0)$, where the functions Φ_1 and Φ_2 are in the space $G(\tau_0)$.

Proof. Suppose that g is an even function of the space $\mathcal{E}(\tau_0)$, i.e. $g(z)$

$= \sum\limits_{n=0}^{\infty} a_n H_{2n}(z),\ z \in S(\tau_0)$. Since $g(\sqrt{z})$ is single-valued, it is holomorphic in the region $\Delta(\tau_0)$ and, due to the relations [Chapter I, (5.24)], it has there a representation by a series in the Laguerre polynomials $\{L_n^{(-1/2)}(z)\}_{n=0}^{\infty}$. Then **(VI.1.4)** gives

$$g(\sqrt{z}) = \exp z \int_0^\infty t^{-1/2} \exp(-t)\tilde{\Phi}_1(t) B_{-1/2}(zt)\, dt,$$

where $\tilde{\Phi}_1 \in G(\tau_0)$. Since $B_{-1/2}(w) = w^{1/4} J_{-1/2}(2\sqrt{w}) = \pi^{-1/2}\cos(2\sqrt{w})$ [Appendix, (2.11), (a)], we have

$$g(z) = 2\exp z^2 \int_0^\infty \exp(-t^2)\Phi_1(t^2)\cos(2zt)\, dt,$$

where $\Phi_1(w) = \pi^{-1/2}\tilde{\Phi}_1(w)$ i.e. $\Phi_1 \in G(\tau_0)$.

Suppose that the function $h \in \mathcal{E}(\tau_0)$ is odd, i.e. $h(z) = \sum\limits_{n=0}^{\infty} b_n H_{2n+1}(z),\ z \in S(\tau_0)$. Then the function $(\sqrt{z})^{-1} h(\sqrt{z})$ is analytic in the simply connected region $\Delta(\tau_0)$, hence, it is holomorphic there. Moreover, due to the relations [Chapter I, (5.25)], for $z \in \Delta(\tau_0)$ it has a series representation by the polynomials $\{L_n^{(1/2)}(z)\}_{n=0}^{\infty}$.

Further, using **(VI.1.4)** as well as that $B_{1/2}(w) = w^{-1/4} J_{1/2}(2\sqrt{w}) = \pi^{-1/2}(\sqrt{w})^{-1}\sin(2\sqrt{w})$ [Appendix, (2.11), (b)] we obtain the representation

$$h(z) = 2i\exp z^2 \int_0^\infty t\exp(-t^2)\Phi_2(t^2)\sin(2zt)\, dt,\ z \in S(\tau_0),$$

where $\Phi_2 \in G(\tau_0)$.

In the general case $f(z) = g(z) + h(z)$, where $g(z) = (f(z) + f(-z))/2$ is even, $h(z) = (f(z) - f(-z))/2$ is odd and, moreover, if f has a representation by a series in Hermite polynomials in $S(\tau_0)$, then so does each of the functions g and h.

Conversely, suppose that the function f has a representation of the form (4.1) in the strip $S(\tau_0)$, where the functions $\Phi_j, j = 1, 2$ are in $G(\tau_0)$. Suppose that $\Phi_j(w) = \sum\limits_{n=0}^{\infty} (n!)^{-1} a_n^{(j)} w^n, j = 1, 2$. Then for each $z \in S(\tau_0)$ we have

$$f(z) = 2\exp z^2 \int_0^\infty \exp(-t^2)\Big\{\sum_{n=0}^{\infty}(n!)^{-1} a_n^{(1)} t^{2n}\Big\}\cos(2zt)\, dt \tag{4.2}$$

$$+2\exp z^2 \int_0^\infty \exp(-t^2)\Big\{\sum_{0}^{\infty}(n!)^{-1} a_n^{(2)} t^{2n+1}\Big\}\sin(2zt)\, dt.$$

Further, exchanging the summations and integrations, and taking [Chapter I, Exercise 30] into consideration, we obtain that f has a representation as a series in Hermite polynomials in the strip $S(\tau_0)$.

4.2 As an application of **(VI.2.1)**, we are able to give a necessary and sufficient condition for a complex function to be in the space $\mathcal{G}^+(\tau_0), 0 \le \tau_0 < \infty$, i.e. to be representable in the half-plane $H^+(\tau_0) : \operatorname{Im} z > \tau_0$ as a series in the Hermite associated functions $\{G_n^+(z)\}_{n=0}^{\infty}$. Indeed, the relations [Chapter I, (5.26), (5.27)], the equalities $K_{1/2}(z) = K_{-1/2}(z) = \sqrt{\pi/2z}\exp(-z)$ [Appendix, (2.13)] and the proposition **(VI.2.1)** lead to the following result:

(VI.4.2) *A complex function f is in the space $\mathcal{G}^+(\tau_0), 0 \le \tau_0 < \infty$, if and only if it has the representation*

$$(4.3) \qquad f(z) = \int_0^\infty \{\Psi_1(t^2) + t\Psi_2(t^2)\} \exp(-t^2 + 2izt)\, dt, \;\; z \in H^+(\tau_0),$$

with $\Psi_1, \Psi_2 \in G(-\tau_0)$.

Remark. The above assertion remains true if we replace the space $\mathcal{G}^+(\tau_0)$ by $\mathcal{G}^-(\tau_0)$, the half-plane $H^+(\tau_0)$ by $H^-(-\tau_0) : \operatorname{Im} z < -\tau_0$ as well as z by $-z$.

5. Representation of entire functions of exponential type by series in Laguerre and Hermite polynomials

5.1 Now we shall see that for entire functions of exponential type it is possible to give sufficient as well as necessary conditions in order that they are representable by series in Laguerre or Hermite polynomials in terms of their indicator functions.

(VI.5.1) *If f is an entire functions of exponential type and $h_f(0) < 1/2$, then $f \in \mathcal{L}^{(\alpha)}(\infty)$ for each $\alpha > -1$.*

Proof. Denote by $B(f; z)$ the Borel transform of f and define the entire function $\Phi_\alpha(f; w)$ by

$$\Phi_\alpha(f; w) = \frac{1}{2\pi i} \int_\gamma (1-\zeta)^{-1-\alpha} \exp\Big(-\frac{w\zeta}{1-\zeta}\Big) B(f; \zeta)\, d\zeta,$$

where γ is a positively oriented Jordan curve which is located in the half-plane $\operatorname{Re}\zeta < 1/2$ and contains the conjugate diagram of the function f in its interior.

It is easily seen that $\Phi_\alpha(f; w)$ is of exponential type and, moreover, that its type is less than one. Therefore, $\Phi_\alpha(f; w) \in G(\infty)$ and then from (1.12) we obtain that for $z \in \mathbb{C}$

$$\exp z \int_0^\infty t^\alpha \exp(-t) \Phi_\alpha(f; t) B_\alpha(zt)\, dt = \frac{1}{2\pi i} \int_\gamma \exp(z\zeta) B(f; \zeta)\, d\zeta = f(z)$$

and **(VI.1.3)** yields immediately $f \in \mathcal{L}^{(\alpha)}(\infty)$.

Remark. Observe that the above assertion is not true if $h_f(0) = 1/2$. Indeed, the entire function $\exp(z/2)$ cannot be represented in the whole complex plane by a (convergent) series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}, \alpha > -1$, since it is not in the space $\mathcal{L}(\infty)$ [**(V.3.7)**]. In fact it is not in the space $\mathcal{L}(\lambda_0)$ for any $\lambda_0 \in (0, \infty)$.

The next proposition could be considered as an inverse of the preceding one.

(VI.5.2) *Suppose that the entire function* Φ, *defined by the equality* (1.3), *is of exponential type and that its type is less than one. If* $\alpha > -1$, *then* (1.8) *defines an entire function of exponential type and, moreover,* $h_f(0) < 1/2$.

Proof. Denote by τ the type of Φ. Since $\tau = \limsup_{n\to\infty} |a_n|^{1/n} < 1$, it follows that $\lim_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n| = -\infty$ and from **(VI.1.2)** we obtain that $\Phi \in G(\infty)$. Then **(VI.1.3)** yields that in the case under consideration (1.8) defines an entire function. If $B(\Phi; w)$ is the Borel transform of Φ, then (1.8) and (1.12) yield that for $\rho \in (\tau, 1)$,

$$f(z) = \frac{1}{2\pi i}\int_{C(0,\rho)} (1-\zeta)^{-1-\alpha}\exp\Big(-\frac{z\zeta}{1-\zeta}\Big)B(\Phi;\zeta)\,d\zeta. \tag{5.1}$$

Denote

$$M(\rho) = \max\{|(1-\zeta)^{-1-\alpha}B(\Phi;\zeta)| : \zeta \in C(0;\rho)\}$$

and

$$\sigma(\rho) = \max\{|\zeta(1-\zeta)^{-1}| : \zeta \in C(0;\rho)\},$$

then from (5.1) it follows that for $z \in \mathbb{C}, |f(z)| \le M(\rho)\exp(\sigma(\rho)|z|)$. Hence, f is of exponential type and, moreover, (5.1) yields the estimate

$$h_f(0) \le \max_{\zeta\in C(0;\rho)} \mathrm{Re}\{-\zeta(1-\zeta)^{-1}\} = \rho(1+\rho)^{-1} < 1/2. \tag{5.2}$$

Remark. Since the inequality (5.2) holds for each $\rho \in (\tau, 1)$, we have in fact $h_f(0) \le \tau(1+\tau)^{-1}$.

5.2 Now we are to pay some attention to the representation of entire functions of exponential type by series in Hermite polynomials. Again, the main tool we are going to use is the Borel transform.

(VI.5.3) *Every entire function* f *of exponential type is representable by a series in the Hermite polynomials in the whole complex plane. Moreover, if* $\{a_n\}_{n=0}^{\infty}$ *are the coefficients of the representation, then*

$$a_n = (n!2^n)^{-1}\sum_{k=0}^{\infty}\frac{f^{(k+n)}(0)}{k!4^k},\ n = 0, 1, 2, \ldots. \tag{5.3}$$

Proof. Denote by C a positively oriented circle with center at the origin which contains the indicator diagram of the function f in its interior. Then, substituting $w = \zeta/2$ in [Chapter II, (3.2)], we find that

$$f(z) = \frac{1}{2\pi i}\int_C \left\{\sum_{n=0}^{\infty}(n!2^n)^{-1}H_n(z)\zeta^n\right\}\exp(\zeta^2/4)B(f;\zeta)\,d\zeta = \sum_{n=0}^{\infty} a_n H_n(z),$$

where $a_n = (n!2^n)^{-1}b_n,\ n = 0,1,2,\dots$ and

$$b_n = \frac{1}{2\pi i}\int_C \zeta^n \exp(\zeta^2/4)B(f;\zeta)\,d\zeta,\ n = 0,1,2,\dots.$$

If τ is the type of f, then from the expansion $B(f;\zeta) = \sum_{k=0}^{\infty} f^{(k)}(0)\zeta^{-k-1}$ which holds in the neighbourhood $\overline{\mathbb{C}} \setminus \overline{U(0;\tau)}$ of the point of infinity, it follows that

$$b_n = \frac{1}{2\pi i}\int_C \zeta^n \exp(\zeta^2/4)\left\{\sum_{k=0}^{\infty} f^{(k)}(0)\zeta^{-k-1}\right\}d\zeta$$

$$= \sum_{k=0}^{\infty}\left\{\frac{1}{2\pi i}\int_C \zeta^{n-k-1}\exp(\zeta^2/4)\,d\zeta\right\}f^{(k)}(0),\ n = 0,1,2,\dots.$$

Since

$$\int_C \zeta^{n-k-1}\exp(\zeta^2/4)\,d\zeta = 0,\ k = 0,1,2,\dots n-1,$$

we find that

$$b_n = \sum_{k=n}^{\infty}\beta_{kn}(k!)^{-1}f^{(k)}(0),\ n = 0,1,2,\dots,$$

where

$$\beta_{kn} = \frac{k!}{2\pi i}\int_C \zeta^{n-k-1}\exp(\zeta^2/4)\,d\zeta,\ n = 0,1,2,\dots; k = n, n+1, n+2, \dots.$$

Further, from the expansion

$$\zeta^{\nu}\exp(\zeta^2/4) = \sum_{s=\nu}^{\infty}\frac{\zeta^{2s-\nu}}{(s-\nu)!4^{s-\nu}},\ \nu = 0,1,2,\dots$$

it follows that $\beta_{kn} = k!/(k-n)!4^{k-n},\ n = 0,1,2,\dots; k = n, n+1, n+2, \dots$ and, thus, we obtain the representations (5.3).

Now, we are going, as a corollary of the above proposition, to give a test for a series in the Hermite polynomials to represent an entire functions of exponential type.

(VI.5.4) *The sum of a series in the Hermite polynomials with coefficients* $\{a_n\}_{n=0}^{\infty}$ *is an entire functions of exponential type not greater that* σ *if and only if*

$$\lambda = \limsup_{n\to\infty} |n!a_n|^{n/2} \le \sigma/2. \tag{5.4}$$

Proof. If f is an entire function of exponential type σ, then for each $\tau \in (\sigma, \infty)$ there exists a positive constant $M = M(\tau)$ such that the inequality $|f^{(k)}(0)| \le M\tau^k$ holds for $k = 0, 1, 2, \ldots$. Then the coefficients representations (5.3) yield $|a_n| \le M(n!2^n)^{-1}\tau^n \exp(\tau/4)$. Hence, $\limsup_{n\to\infty} |n!a_n|^{1/n} \le \tau/2$ and since $\tau \in (\sigma, \infty)$ is arbitrary, (5.4) follows.

Conversely, suppose that (5.4) holds. If $\rho \in (\sigma/2, \infty)$, then there exists a positive integer $n_0 = n_0(\rho)$ such that $|a_n| \le (n!)^{-1}\rho^n$ for $n > n_0$. Since $n! \ge (n/e)^n$, we find that $(2n/e)^{n/2}|a_n| \le (2e\rho^2/n)^{n/2}$ and, hence, $\lim_{n\to\infty}(2n/e)^{n/2}|a_n| = 0$. From **(II.3.1)** it follows that the series in the Hermite polynomials with coefficients $\{a_n\}_{n=0}^{\infty}$ converges in the whole complex plane. Let f be its sum and let φ be the sum of the Maclaurin series with coefficients $\{k!a_k\}_{k=0}^{\infty}$. Then for $z \in \mathbb{C}$ we find that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}(n!)^{-1}H_n(z)\left\{\sum_{k=0}^{\infty} k!a_k(2\pi i)^{-1}\int_{C(0;\rho)} \zeta^{n-k-1}\,d\zeta\right\} \\ &= (2\pi i)^{-1}\int_{C(;\rho)}\left\{\sum_{k=0}^{\infty} k!a_k\zeta^{-k-1}\right\}\left\{\sum_{n=0}^{\infty}(n!)^{-1}H_n(z)\zeta^n\right\} d\zeta \end{aligned} \tag{5.5}$$

and, having in mind [Chapter II, (3.2)], we obtain the representation

$$f(z) = (2\pi i)^{-1}\int_{C(0;\rho)} \exp(-\zeta^2 + 2z\zeta)\varphi(\zeta^{-1})\zeta^{-1}\,d\zeta. \tag{5.6}$$

Notice that both series in the second row of (5.5) are uniformly convergent on the circle $C(0;\rho)$ and, hence, the exchange of summation and integration is possible.

Further, from (5.6), it follows that f is of exponential type not exceeding 2ρ. Since $\rho \in (\sigma/2, \infty)$ is arbitrary, we find that f is of type not greater than σ.

Exercises

1. Suppose that $0 \le \tau_0 < \infty$ and $f \in \mathcal{G}^*(\tau_0)$. If $\tau > \tau_0$, then the asymptotic expansion

$$f(z) \sim \sum_{n=0}^{\infty} \frac{c_n}{z^{n+1}},\ z \to \infty$$

holds in the closed half-planes $\overline{H^+(\tau)}$, where

$$\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log\left|\sum_{k=0}^{n}(-1)^k\frac{c_{2k}}{\Gamma(k+1/2)}\right|\leq\tau_0$$

and

$$\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log\left|\sum_{k=0}^{n}(-1)^k\frac{c_{2k+1}}{\Gamma(k+3/2)}\right|\leq\tau_0.$$

Conversely, if the above inequalities hold and, moreover, if for each $\tau>\tau_0$ it is satisfied the equality

$$\lim_{z\in\overline{H^+(\tau)},z\to\infty} z^n\left\{f(z)-\sum_{k=0}^{n-1}\frac{c^k}{z^{k+1}}\right\}=0$$

for the function $f\in\mathcal{H}(H^+(\tau_0))$ uniformly for $n\geq n_0(\tau)$, then $f\in\mathcal{G}(\tau_0)$.

2. Justify the exchange of the summations and integrations in (4.2).

3. As an application of **(VI.2.1)** prove that if $z\in\mathbb{C}\setminus(-\infty,0],\zeta\in\mathbb{C}\setminus[0,\infty)$ and $\operatorname{Re}z^{1/2}>\operatorname{Re}(-\zeta)^{1/2}$, then

$$-2z^{-\alpha/2}(-\zeta)^{\alpha/2}\int_0^\infty J_\alpha(2\sqrt{zt})K_\alpha(2\sqrt{-\zeta t})\,dt=\frac{1}{\zeta-z}$$

provided $\alpha>-1$.

4. Suppose that $0\leq\mu_0<\infty$ and $\alpha\in\mathbb{R}\setminus\mathbb{Z}^-$. Define the mapping $M_{\mu_0,\alpha}$ of the space $\mathcal{M}^{(\alpha)}(\mu_0)$ in the space $G(-\mu_0)$ assuming that to a function f represented in the region $\Delta^*(\mu_0)$ by the series [II, (2.5)] there corresponds the entire function (2.2). Prove that $M_{\mu_0,\alpha}$ is an (algebraic) isomorphism of $\mathcal{M}^{(\alpha)}(\mu_0)$ and $G(-\mu_0)$ considered as $\mathbb{C}$-vector spaces.

5. Suppose that $0\leq\mu_0<\infty$ and $\alpha\in\mathbb{R}\setminus\mathbb{Z}^-$. Define the mapping $L_{\mu_0,\alpha}$ of the space $\mathcal{M}^{(\alpha)}(\mu_0)$ into the space $\mathcal{B}(\mu_0,\alpha)$, assuming that to a function f represented by the series [Chapter IV, (2.5)] there corresponds the function F defined by (3.5). Prove that $L_{\mu_0,\alpha}$ is an (algebraic) isomorphism of $\mathcal{M}^{(\alpha)}(\mu_0)$ and $\mathcal{B}(\mu_0,\alpha)$ considered as $\mathbb{C}$-vector spaces.

6. Denote by $\tilde{G}(\tau_0), 0<\tau_0\leq\infty$, the set of all entire functions of the kind $\tilde{\Phi}(w)=\Phi_1(w^2)+w\Phi_2(w^2)$ provided Φ_1 and Φ_2 are in $G(\tau_0)$. Verify the validity of the proposition that (4.1) establishes an (algebraic) isomorphism of $\mathcal{E}(\tau_0)$ and $\tilde{G}(\tau_0)$ considered as $\mathbb{C}$-vector spaces.

Comments and References

The idea to "engage" classical integral transforms in the problem of representation by series in Laguerre and Hermite polynomials is not new. It has been used e.g. in a paper of Ervin Feldheim [1].

The results, given both as propositions and as exercises, show that the vector spaces of holomorphic functions which are representable e.g. by series in Laguerre polynomials or in Laguerre associated functions, are isomorphic, *via* integral transforms of Hankel or Meijer type, to suitable vector spaces of entire functions of exponential type.

The "key Lemma" **(VI.1.2)** as well as the proposition **(VI.1.3)** can be found in in [P. Rusev, 8, 9, 10, 24, 25] and **(VI.1.4)** is due to L. Boyadjiev [2].

The proposition **(VI.2.4)** as well as that given as Exercise 1 is due to I. Kasandrova [1]. It provides a "growth" characteristic of the functions in the space $\mathcal{M}^{(\alpha)}(\mu_0), -1/2 \leq \alpha \leq 1/2, 0 \leq \mu_0 < \infty$.

We note that **(VI.2.2)** is a "slight" modification of a Lemma due to T. Carleman [1, p.16, I]. The proposition (VI.2.3) is given in [E. Rieckstiņš, 1, p. 63, Theorems 4.3, 4.3′].

The proof of **(VI.3.1)** can be found in [P. Rusev, 26] and that of the "multiplication rule" for the spaces of the kind $\mathcal{M}^{(\alpha)}(\mu_0)$ is published in [P. Rusev, 19].

The proposition **(VI.5.1)** is proved in [R.P. Boas and R.C. Buck, 1, p. 40, (X)]. Its "converse", i.e. **(VI.5.2)** is due to L. Boyadjiev [3]. The coefficient's criterion in order that a series in the Hermite polynomials to represent an entire function of exponential type can be found in [L. Boyadjiev, 4].

Chapter VII

BOUNDARY PROPERTIES OF SERIES IN JACOBI, LAGUERRE AND HERMITE SYSTEMS

1. Convergence on the boundaries of convergence regions

1.1 The asymptotic formulas for the Jacobi systems as well as the corresponding formula of Christoffel-Darboux type for these systems allow to study the convergence of series in Jacobi polynomials and Jacobi associated functions on the boundaries of their regions of convergence. This can be realized, as in the classical theory of Fourier series, by means of appropriate asymptotic formulas for their partial sums.

Here we restrict our considerations to Jacobi series representations of functions in the spaces $\mathcal{H}(E(r)), \mathcal{H}(E^*(r)), 1 < r < \infty$, defined by means of Cauchy type integrals.

The following auxiliary proposition is an analogon of the classical Riemann-Lebesgue Lemma.

(VII.1.1) *Suppose that* $-\infty < A < a < b < B < \infty$ *and* $-\infty < \theta_1 \le \theta_2 < \infty$ *are such that* $\theta_1 \le A - a$ *and* $\theta_2 \le B - b$. *Let* $h(t), A \le t \le B$, *be an L-integrable function and let* $g(t,\theta), a \le t \le b, \theta_1 \le \theta \le \theta_2$, *be a bounded function with bounded variation in the interval* $[a,b]$ *for each fixed* $\theta \in [\theta_1, \theta_2]$. *Then*

$$\lim_{|k|\to\infty} \int_{t_1}^{t_2} h(t \pm \theta) g(t,\theta) \exp ikt\, dt = 0 \tag{1.1}$$

uniformly with respect to $t_1, t_2 \in [a,b]$ *and* $\theta \in [\theta_1, \theta_2]$.

Proof. It is sufficient to prove the assertion in the case when h and g are real. Suppose, in addition, that h is absolutely continuous. Then

$$\lim_{|k|\to\infty} \int_{t_1}^{t_2} h(t \pm \theta) \exp ikt\, dt = 0 \tag{1.2}$$

uniformly with respect to $t_1, t_2 \in [a,b]$, $\theta \in [\theta_1, \theta_2]$ and this can be proved by integration by parts.

Further, we apply the second mean value theorem to the real part of the integral on the left-hand side of (1.1). Thus, we obtain

$$R_k(\theta; t_1, t_2) = \int_{t_1}^{t_2} h(t \pm \theta) g(t,\theta) \cos kt\, dt$$

$$= g(t_1, \theta) \int_{t_1}^{\tau} h(t \pm \theta) \cos kt\, dt + g(t_2, \theta) \int_{\tau}^{t_2} h(t \pm \theta) \cos kt\, dt.$$

If $K = \sup\{|g(t,\theta)| : t \in [a,b], \theta \in [\theta_1,\theta_2]\}$, then

$$|R_k(\theta; t_1, t_2)| \le K\left\{\left|\int_{t_1}^{\tau} h(t \pm \theta)\cos kt\, dt\right| + \left|\int_{\tau}^{t_2} h(t \pm \theta)\cos kt\, dt\right|\right\},$$

and (1.2) yields that $\lim_{|k|\to\infty} R_k(\theta; t_1, t_2) = 0$ uniformly with respect to $t_1, t_2 \in [a,b]$ and $\theta \in [\theta_1,\theta_2]$. In the same way, we prove that

$$\lim_{|k|\to\infty} I_k(\theta; t_1, t_2) = \lim_{|k|\to\infty} \int_{t_1}^{t_2} h(t \pm \theta) g(t,\theta) \sin kt\, dt = 0$$

uniformly with respect to $t_1, t_2 \in [a,b]$ and $\theta \in [\theta_1,\theta_2]$.

If h is L-integrable and $\varepsilon > 0$, then there exists an absolutely continuous function $\tilde{h}(t), A \le t \le B$ such that

$$\int_A^B |h(t) - \tilde{h}(t)|\, dt < \varepsilon(1+K)^{-1}.$$

Hence, the inequality

$$\left|\int_{t_1}^{t_2} \{h(t \pm \theta) - \tilde{h}(t \pm \theta)\} g(t,\theta) \exp ikt\, dt\right| < \varepsilon$$

holds for $t_1, t_2 \in [a,b], \theta \in [\theta_1,\theta_2]$ and $|k| = 0, 1, 2, \ldots$. It yields that (1.1) is true uniformly with respect to $t_1, t_2 \in [a,b]$ and $\theta \in [\theta_1,\theta_2]$.

(VII.1.2) *Suppose that $\alpha+1, \beta+1$ and $\alpha+\beta+2$ are not equal to $0, -1, -2, \ldots$. Let $\varphi(\zeta)$ be an L-integrable complex-valued function on the ellipse $e(r)$, $1 < r < \infty$, and denote by $\{S_\nu^{(\alpha,\beta)}(z)\}_{\nu=0}^\infty$ the partial sums of the series in Jacobi polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^\infty$ which represents the function* [Chapter V, (1.1)] *in the region $E(r)$.*

Then there exists a sequence of complex-valued functions $\{r_\nu^{(\alpha,\beta)}(z)\}_{\nu=0}^\infty$ which are holomorphic in the region $\mathbb{C} \setminus [-1,1]$, and such that

$$S_\nu^{(\alpha,\beta)}(z) = \frac{1}{2\pi i}\int_{e(r)} \frac{1 - (\omega(z)/\omega(\zeta))^\nu}{\zeta - z}\varphi(\zeta)\, d\zeta + r_\nu^{(\alpha,\beta)}(z),\ z \in \mathbb{C} \setminus [-1,1]. \tag{1.3}$$

Moreover, $\lim_{\nu\to\infty} r_\nu^{(\alpha,\beta)}(z) = 0$ uniformly on $e(r)$.

Proof. Denote by G the region $\mathbb{C} \setminus [-1,1]$. From [Chapter I, (4.28)], [Chapter V, (1.1)] and [Chapter V, (1.2)] it follows that if $z \in G \setminus e(r)$ and $\nu = 0, 1, 2, \ldots$, then

$$S_\nu^{(\alpha,\beta)}(z) = \frac{1}{2\pi i}\int_{e(r)} \frac{1 - \Delta_\nu^{(\alpha,\beta)}(z,\zeta)}{\zeta - z}\varphi(\zeta)\, d\zeta. \tag{1.4}$$

Since $\Delta_\nu^{(\alpha\beta)}(z,z) \equiv 1$ for $z \in G$ and $\nu = 0,1,2,\dots$ [Chapter I, Exercise 27, (a)], $\{1 - \Delta_\nu^{(\alpha,\beta)}(z,\zeta)\}(\zeta - z)^{-1}$ as a function of $\zeta \in G \setminus \{z\}$ has a removable singularity at the point $\zeta = z$ and, moreover, the representation (1.4) holds in the whole region G.

If $(z,\zeta) \in G\times G$, then from the asymptotic formulas [Chapter III,(1.9)], [Chapter III,(1.30)] and from Stirling's formula we obtain that

$$(1.5) \qquad \Delta_\nu^{(\alpha,\beta)}(z,\zeta) = (\omega(z)/\omega(\zeta))^\nu\{D^{(\alpha,\beta)}(z,\zeta) + \delta_\nu^{(\alpha,\beta)}(z,\zeta)\},\ \nu = 1,2,3,\dots,$$

where $D^{(\alpha,\beta)}(z,\zeta)$ and $\{\delta_\nu^{(\alpha,\beta)}(z,\zeta)\}_{\nu=1}^\infty$ are complex-valued functions which are holomorphic in the region $G \times G$. Moreover,

$$(1.6) \qquad \lim_{\nu\to\infty} \delta_\nu^{(\alpha,\beta)}(z,\zeta) = 0$$

uniformly on each compact subset of this region.

From (1.5) it follows that $D^{(\alpha,\beta)}(z,z) + \delta_\nu^{(\alpha,\beta)}(z,z) = 1$ for every $z \in G$ and $\nu = 1,2,3,\dots$. Then, having in mind (1.6), we obtain that $D^{(\alpha,\beta)}(z,z) \equiv 1$ in G and, hence, $\delta_\nu^{(\alpha,\beta)}(z,z) \equiv 0$ in G for each $\nu = 1,2,3,\dots$. Assuming that $r_0^{(\alpha,\beta)}(z) = S_0^{(\alpha,\beta)}(z)$ for $z \in G$, we obtain the representation (1.3) with $r_\nu^{(\alpha,\beta)}(z) = r_{\nu,1}^{(\alpha,\beta)}(z) + r_{\nu,2}^{(\alpha,\beta)}(z),\ \nu = 1,2,3,\dots$, where

$$(1.7) \qquad r_{\nu,1}^{(\alpha,\beta)}(z) = \frac{1}{2\pi i}\int_{e(r)} \frac{1 - D^{(\alpha,\beta)}(z,\zeta)}{\zeta - z}(\omega(z)/\omega(\zeta))^\nu \varphi(\zeta)\,d\zeta,\ \nu = 1,2,3,\dots$$

and

$$(1.8) \qquad r_{\nu,2}^{(\alpha,\beta)}(z) = \frac{1}{2\pi i}\int_{e(r)} \frac{\delta_\nu^{(\alpha,\beta)}(z,\zeta)}{\zeta - z}(\omega(z)/\omega(\zeta))^\nu \varphi(\zeta)\,d\zeta,\ \nu = 1,2,3,\dots.$$

If $0 < \tau < d(r)$ =dist$(e(r),[-1,1])$, then we define $M(\tau) = \cup_{z\in e(r)}\overline{U(z;\tau)}$. Obviously, $M(\tau)$ is a compact subset of the region G and $e(r) \subset M(\tau)$. Further, we define $m_\nu(\tau) = \max\{|\delta_\nu^{(\alpha,\beta)}(z,\zeta)| : (z,\zeta) \in e(r) \times M(\tau)\}$. Then Schwarz's Lemma gives $|\delta_\nu^{(\alpha,\beta)}(z,\zeta)(\zeta - z)^{-1}| \le m_\nu(\tau)\tau^{-1}$ provided $(z,\zeta) \in e(r) \times M(\tau)$ and $\nu = 1,2,3,\dots$.

If $u(t) = (r\exp it + r^{-1}\exp(-it))/2$ and $\tilde\varphi(t) = \varphi(u(t)),\ t \in \mathbb{R}$, then substituting $\zeta = u(t), -\pi \le t \le \pi$ in the integral in (1.8), and taking into account that $|\omega(z)/\omega(\zeta)| = 1$ for $(z,\zeta) \in e(r) \times e(r)$, we obtain that

$$|r_{\nu,2}^{(\alpha,\beta)}(z) \le (2\pi\tau)^{-1}m_\nu(\tau)\int_{-\pi}^{\pi} |\tilde\varphi(t)u'(t)|\,dt,\ \nu = 1,2,3,\dots$$

for $z \in e(r)$. Hence, $\lim_{\nu\to\infty} r_{\nu,2}^{(\alpha)}(z) = 0$ uniformly on $e(r)$.

If $(z,\zeta)\in G\times G$, then we define $\tilde{D}^{(\alpha,\beta)}(z,\zeta)=(1-D^{(\alpha,\beta)}(z,\zeta))(\zeta-z)^{-1}$ when $z\neq\zeta$ and assume that $\tilde{D}^{(\alpha,\beta)}(z,z)=\{\partial D^{(\alpha,\beta)}(z,\zeta)/\partial\zeta\}_{\zeta=z}$. It is easy to see that this function is holomorphic in $G\times G$.

We set $\omega(z)/\omega(\zeta)=\exp(-it), -\pi\leq t\leq\pi$, in the integral (1.7) and since $\zeta=u(t+\theta)$ for $z=u(\theta), -\pi\leq\theta\leq\pi$, we find that

$$r_{\nu,1}^{(\alpha,\beta)}(z)=\int_{-\pi}^{\pi}\tilde{\varphi}(t+\theta)\tilde{g}(t,\theta)\exp(-i\nu t)\,dt,\ \nu=1,2,3,\dots,$$

where $\tilde{g}^{(\alpha,\beta)}(t,\theta)=(2\pi i)^{-1}\tilde{D}^{(\alpha,\beta)}(u(\theta),u(t+\theta))u'(t+\theta)$.

If $\theta\in[-\pi,\pi]$ is fixed, then $\tilde{g}^{(\alpha,\beta)}(t,\theta)$ as a function of $t\in\mathbb{R}$ is in the class $\mathcal{C}^{\infty}$. Hence, it has a bounded variation in the interval $-\pi\leq t\leq\pi$. Moreover, as a function of the variables t,θ, it is bounded for $(t,\theta)\in\mathbb{R}\times[-\pi,\pi]$. Then **(VII.1.1)** yields that

$$\lim_{\nu\to\infty}\int_{-\pi}^{\pi}\tilde{\varphi}(t+\theta)\tilde{g}^{(\alpha,\beta)}(t,\theta)\exp(-i\nu t)\,dt=0$$

uniformly with respect to $\theta\in[-\pi,\pi]$, i.e. $\lim_{\nu\to\infty}r_{\nu,1}^{(\alpha,\beta)}(z)=0$ uniformly on $e(r)$.

We define $e(z,\tau)=e(r)\cap\overline{U(z;\tau)}$ for $z\in e(r)$ and $0<\tau<d(r)$, i.e $e(z,\tau)$ is that arc of the ellipse $e(r)$ which is contained in the closed disk $\overline{U}(z;\tau)$. We suppose that this arc is positively oriented and denote by $z_{1,\tau}$ and $z_{2,\tau}$ its initial and end-point, correspondingly. By setting $\omega(z)/\omega(\zeta)=\exp(-it)$, the parametric equation of the ellipse $e(r)$ becomes $\zeta=u(t+\theta), -\pi\leq t\leq\pi$. Since the point $z=u(\theta), -\pi\leq\theta\leq\pi$ corresponds to $t=0$, we have $z_{j,\tau}=u(t_{j,\tau}+\theta), j=1,2$, where $-\pi<t_{1,\tau}<0<t_{2,\tau}<\pi$.

(VII.1.3) *Suppose that* $\alpha+1,\beta+1$ *and* $\alpha+\beta+2$ *are not equal to* $0,-1,-2,\dots$. *Let* $\tilde{e}(r)\subset e(r), 1<r<\infty$, *be a closed arc* (*which may reduce to a point*) *and let* $\varphi(\zeta)$ *be an L-integrable* (*complex-valued*) *function on* $e(r)$ *such that:*

(a) φ *is bounded on* $\tilde{e}(r)$;

(b) *for each* $\varepsilon>0$ *there exists* $\tau=\tau(\varepsilon)\in(0,d(r))$ *such that*

$$\int_{e(z,\tau)}\left|\frac{\varphi(\zeta)-\varphi(z)}{\zeta-z}\right|dm(r)<\varepsilon$$

for $z\in\tilde{e}(r)$, *where* $m(r)$ *is the Lebesgue measure on the ellipse* $e(r)$.

Then the series in Jacobi's polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ *which represents the function* [V, (1.1)] *in the region* $E(r)$ *is uniformly convergent on the arc* $\tilde{e}(r)$ *and*

$$\frac{1}{2\pi i}\int_{e(r)}\frac{\varphi(\zeta)}{\zeta-z}\,d\zeta+\frac{1}{2}\varphi(z)\tag{1.9}$$

is its sum for $z\in\tilde{e}(r)$.

Proof. From the assumption **(b)** it follows that if $z \in \tilde{e}(r)$, then the integral

$$\frac{1}{2\pi i}\int_{e(r)} \frac{\varphi(\zeta)-\varphi(z)}{\zeta - z}\, d\zeta$$

exists as a main value in the Cauchy sence. Since in the same sence we have

$$\frac{1}{2\pi i}\int_{e(r)} \frac{d\zeta}{\zeta - z} = \frac{1}{2},\ z \in e(r),$$

the integral

$$\frac{1}{2\pi i}\int_{e(r)} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}\int_{e(r)} \frac{\varphi(\zeta)-\varphi(z)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i}\int_{e(r)} \frac{\varphi(z)}{\zeta - z}\, d\zeta,\ z \in \tilde{e}(r)$$

also exists in the Cauchy sence. Then from (1.3) it follows that for $z \in \tilde{e}(r)$ and $\nu = 0, 1, 2, \dots$

$$S_\nu^{(\alpha,\beta)}(z) = \frac{1}{2\pi i}\int_{e(r)} \frac{\varphi(\zeta)-\varphi(z)}{\zeta - z}\, d\zeta + \frac{\varphi(z)}{2\pi i}\int_{e(r)} \frac{1-(\omega(z)/\omega(\zeta))^\nu}{\zeta - z}\, d\zeta$$
$$-\frac{1}{2\pi i}\int_{e(r)} \frac{\varphi(\zeta)-\varphi(z)}{\zeta - z}(\omega(z)/\omega(\zeta))^\nu\, d\zeta + r_\nu^{(\alpha,\beta)}(z).$$

The point z is a removable singularity for each of the functions

$$\{1-(\omega(z)/\omega(\zeta))^\nu\}(\zeta - z)^{-1}, \nu = 0, 1, 2, \dots,$$

considered as functions of ζ in the region $\{G \setminus \{z\}\} \cup \{\infty\}$. Hence, the Cauchy integral theorem yields that for $R > (r + r^{-1})/2$ and $\nu = 1, 2, 3, \dots,$

$$\frac{1}{2\pi i}\int_{e(r)} \frac{1-(\omega(z)/\omega)(\zeta))^\nu}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}\int_{C(R)} \frac{1-(\omega(z)/\omega(\zeta))^\nu}{\zeta - z}\, d\zeta,$$

where $C(R)$ is the positively oriented circle centered at the origin and with radius R.

Since

$$\lim_{R\to\infty}\int_{C(R)} \frac{1-(\omega(z)/\omega(\zeta))^\nu}{\zeta - z}\, d\zeta = 2\pi i,\ \nu = 1, 2, 3, \dots,$$

we obtain that

$$\frac{1}{2\pi i}\int_{e(r)} \frac{1-(\omega(z)/\omega(\zeta))^\nu}{\zeta - z}\, d\zeta = 1,\ \nu = 1, 2, 3, \dots. \tag{1.10}$$

Hence, for $z \in \tilde{e}(r)$ and $\nu = 1, 2, 3, \dots$ we have that

$$S_\nu^{(\alpha,\beta)}(z) = \frac{1}{2\pi i}\int_{e(r)} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2}\varphi(z) \tag{1.11}$$

$$-\frac{1}{2\pi i}\int_{e(r)}\frac{\varphi(\zeta)-\varphi(z)}{\zeta-z}(\omega(z)/\omega(\zeta))^{\nu}\,d\zeta+r_{\nu}^{(\alpha,\beta)}(z).$$

Define

$$I_{\nu,1}(z)=\int_{e(z,\tau)}\frac{\varphi(\zeta)-\varphi(z)}{\zeta-z}(\omega(z)/\omega(\zeta))^{\nu}\,d\zeta$$

and

$$I_{\nu,2}(z)=\int_{e(r)\setminus e(z,\tau)}\frac{\varphi(\zeta)-\varphi(z)}{\zeta-z}(\omega(z)/\omega(\zeta))^{\nu}\,d\zeta$$

provided $z\in\tilde{e}(r)$ and $\nu=1,2,3,\ldots$.

If $\varepsilon>0$, then the assumption **(b)** yields that there exists $\tau=\tau(\varepsilon)$ such that $|I_{\nu,1}(z)|<\varepsilon$ for $z\in\tilde{e}(r)$ and $\nu=1,2,3,\ldots$.

Further, substituting $\omega(z)/\omega(\zeta)=\exp(-it)$, $-\pi\le t\le\pi$, in the integral $I_{\nu,2}(z)$, we obtain

$$I_{\nu,2}(z)=\left(\int_{-\pi}^{t_{1,\tau}}+\int_{t_{2,\tau}}^{\pi}\right)\tilde{\varphi}(t+\theta)g(t,\theta)\exp(-i\nu t)\,dt$$

$$-\varphi(z)\left(\int_{-\pi}^{t_{1,\tau}}+\int_{t_{2,\tau}}^{\pi}\right)g(t,\theta)\exp(-i\nu t)\,dt,$$

where $g(t,\theta)=\{u(t+\theta)-u(\theta)\}^{-1}u'(t+\theta)$.

Since the arc $\tilde{e}(r)$ is a compact set and $t_{j,\tau}$, $j=1,2$ as functions of $z\in\tilde{e}(r)$ are continuous, it follows that $t_{\tau}^{(1)}=\sup t_{1,\tau}$ is negative and $t_{\tau}^{(2)}=\inf t_{2,\tau}$ is positive. Let $z_j=u(\theta_j), j=1,2$ be the end points of the arc $\tilde{e}(r)$. It is clear that the function $g(t,\theta)$ is bounded for $t\in\{[-\pi,t_{\tau}^{(1)}]\cup[t_{\tau}^{(2)},\pi]\}$ and $\theta\in[\theta_1,\theta_2]$. Moreover, if θ is fixed, then this function has bounded variation on each of the intervals $[-\pi,t_{\tau}^{(1)}],[t_{\tau}^{(2)},\pi]$. Then **(VII.1.1)** yields that there exists a positive integer $\nu_0=\nu_0(\varepsilon)$ such that $|I_{\nu,2}(z)|<\varepsilon$ for $z\in\tilde{e}(r)$ and $\nu>\nu_0$. Thus, we find that

$$\lim_{\nu\to\infty}\int_{e(r)}\frac{\varphi(\zeta)-\varphi(z)}{\zeta-z}(\omega(z)/\omega(\zeta))^{\nu}\,d\zeta=0$$

uniformly on the arc $\tilde{e}(r)$ and the assertion we have to prove is a corollary of (1.11).

We say that the function φ satisfies Hölder condition with exponent $\gamma\in(0,1]$ on the arc $\tilde{e}(r)$ if there exist constants K and $\tau\in(0,d(r))$ such that $|\varphi(\zeta)-\varphi(z)|\le K|\zeta-z|^{\gamma}$ for $z\in\tilde{e}(r)$ and $\zeta\in e(z,\tau)$. It is easily seen that in such a case the condition **(b)** of **(VII.1.3)** is satisfied. Now we can formulate the following proposition:

(VII.1.4) *Suppose that $\alpha+1,\beta+1$ and $\alpha+\beta+2$ are not equal to $0,-1,-2,\ldots,1<r<\infty$, and let f be a complex-valued function continuous on the closed set $\overline{E(r)}$ and holomorphic in $E(r)$. If f satisfies Hölder's condition on the arc $\tilde{e}(r)\subset e(r)$, then the series* [IV, (1.2)] *in the Jacobi polynomials*

$\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ *which represents the function* f *in* $E(r)$ *is uniformly convergent on* $\tilde{e}(r)$ *with sum* $f(z)$ *for* $z \in \tilde{e}(r)$. *In particular, if* $\tilde{e}(r) = e(r)$, *then this series is uniformly convergent on* $\overline{E(r)}$.

Proof. The generalized Cauchy integral formula yields that

$$f(z) = \frac{1}{2\pi i}\int_{e(r)} \frac{f(\zeta)}{\zeta - z}\,d\zeta,\ z \in E(r).$$

Moreover, it holds the equality

$$\frac{1}{2\pi i}\int_{e(r)} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2}f(z)$$

for $z \in e(r)$. Then as a corollary of **(VII.1.3)** we obtain that the series [Chapter IV, (1.2)] is uniformly convergent on the arc $\tilde{e}(r)$ and that $f(z)$ is its sum for $z \in \tilde{e}(r)$.

If $\tilde{e}(r) = e(r)$, then the uniform convergence of the series [IV, (1.2)] on $e(r)$ and the maximum modulus principle imply its uniform convergence in the region $E(r)$.

1.2 The possibility to prove assertions about the convergence of series in Jacobi polynomials on the boundaries of their regions of convergence is due to the fact that the coefficients of these series have integral representations in terms of the Jacobi associated functions. Let us note that the last property is a direct corollary of the Christoffel-Darboux formula for the Jacobi systems and of the Cauchy integral theorem as well.

In the case of series in Laguerre and Hermite polynomials the situation is rather different. If we want to use integral representations of their coefficients in terms of the corresponding associated functions *via* formulas of Christoffel-Darboux type, then we have to confine ourselves to classes of holomorphic functions having Cauchy type integral representations in regions of the kind $\Delta(\lambda_0), 0 < \lambda_0 \le \infty$, and $S(\tau_0), 0 < \tau_0 \le \infty$, respectively.

We define $\eta(z,\zeta) = (-z)^{1/2} - (-\zeta)^{1/2}$ and $E_\nu(z,\zeta) = \exp\{2\eta(z,\zeta)\sqrt{\nu+1}\}, \nu = 0, 1, 2, \dots$ provided $z, \zeta \in \mathbb{C} \setminus [0,\infty)$.

(VII.1.5) *Suppose that* $0 < \lambda < \infty, \alpha \in \mathbb{R}\backslash\mathbb{Z}^-$, *and that* F *is a complex-valued function which is L-integrable on the parabola* $p(\lambda)$ *and satisfies the condition* [V, (3.27)].

If $\{S_\nu^{(\alpha)}(z)\}_{\nu=0}^{\infty}$ *are the partial sums of the series in the Laguerre polynomials* $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ *representing the function* [Chapter V, (3.28)] *in the region* $\Delta(\lambda)$, *and* $\{r_\nu^{(\alpha)}(z)\}_{\nu=0}^{\infty}$ *is the sequence defined for* $z \in p(\lambda)$ *by the equalities*

$$(1.12)\qquad S_\nu^{(\alpha)}(z) = \frac{1}{2\pi i}\int_{p(\lambda)} \frac{1 - E_\nu(z,\zeta)}{\zeta - z}F(\zeta)\,d\zeta + r_\nu^{(\alpha)}(z),\ \nu = 0, 1, 2, \dots,$$

then $\lim_{\nu\to\infty} r_\nu^{(\alpha)}(z) = 0$ *uniformly on each finite arc of* $p(\lambda)$.

Proof. First we note that $\zeta = z$ is a removable singularity of the function $E_\nu(z,\zeta)$ for each $\nu = 0, 1, 2, \dots$. From [Chapter V, (3.29)], [Chapter V, (3.30)] and the Christoffel-Darboux formula for the Laguerre systems we obtain that for $z \in p(\lambda)$

$$S_\nu^{(\alpha)}(z) = \frac{1}{2\pi i}\int_{p(\lambda)} \frac{1-\Delta_\nu^{(\alpha)}(z,\zeta)}{\zeta - z} F(\zeta)\,d\zeta,\ \ \nu = 0,1,2,\dots. \tag{1.13}$$

Suppose that $\tilde{p}(\lambda)$ is a finite (closed) arc of the parabola $p(\lambda)$ and define $m(\tilde{p}(\lambda)) = \max\{|z|; z \in \tilde{p}(\lambda)\}$. If $\rho > \max(1, 2\lambda^2, m(\tilde{p}(\lambda)))$, then we denote by $\Gamma(\lambda,\rho)$ the arc of $p(\lambda)$ lying in the disk $\overline{U}(0;\rho)$. The asymptotic formula [Chapter III, (2.3)], the inequality [Chapter III, (5.1)], Stirling's formula as well as the requirement [Chapter V, (3.28)] lead to the conclusion that

$$\lim_{\rho\to\infty} \int_{p(\lambda)\setminus\Gamma(\lambda,\rho)} \frac{1-\Delta_\nu^{(\alpha)}(z,\zeta)}{\zeta - z} F(\zeta)\,d\zeta = 0 \tag{1.14}$$

uniformly for $z \in \tilde{p}(\lambda)$.

Since $|E_\nu(z,\zeta)| = 1$ for $z, \zeta \in p(\lambda)$ and $\nu = 0, 1, 2, \dots$, we have

$$\lim_{\rho\to\infty} \int_{p(\lambda)\setminus\Gamma(\lambda,\rho)} \frac{1-E_\nu(z,\zeta)}{\zeta - z} F(\zeta)\,d\zeta = 0 \tag{1.15}$$

uniformly for $z \in \tilde{p}(\lambda)$.

The validity of the asymptotic formula (1.12) will be established if we show that

$$\lim_{\nu\to\infty}\left\{ S_\nu^{(\alpha)}(z) - \frac{1}{2\pi i}\int_{p(\lambda)} \frac{1-E_\nu(z,\zeta)}{\zeta - z} F(\zeta)\,d\zeta \right\} = 0$$

uniformly for $z \in \tilde{p}(\lambda)$.

If $m(\tilde{p}) = \max\{|z| : z \in \tilde{p}\}$ and $\rho > \max(1, 2\lambda^2, m(\tilde{p}))$, then we define for $z \in p(\lambda)$ and $\nu = 0, 1, 2, \dots$

$$D_\nu^{(\alpha)}(\rho; z) = \frac{1}{2\pi i}\int_{\Gamma(\lambda,\rho)} \frac{\Delta_\nu^{(\alpha)}(z,\zeta) - E_\nu(z,\zeta)}{\zeta - z} F(\zeta)\,d\zeta. \tag{1.16}$$

Therefore, in view of (1.15) and (1.16), it is sufficient to prove that if ρ $> \max(1, 2\lambda^2, m(\tilde{p}))$, then

$$\lim_{\nu\to\infty} D_\nu^{(\alpha)}(\rho; z) = 0 \tag{1.17}$$

uniformly for $z \in \tilde{p}(\lambda)$.

Using the relations [Chapter IV, (2.6)] and [Chapter III, (2.10)] we obtain the representation $(z, \zeta \in \mathbb{C} \setminus [0,\infty), \nu = 1, 2, 3, \dots)$

$$\Delta_\nu^{(\alpha)}(z,\zeta) = -\frac{\Gamma(\nu+2)}{\Gamma(\nu+\alpha+1)}\{(-\zeta)L_{\nu+1}^{\alpha}(z)M_{\nu+1}^{(\alpha-1)}(\zeta) - L_{\nu+1}^{(\alpha)}(z)M_{\nu+1}^{(\alpha)}(\zeta)\}.$$

Further, from the asymptotic formulas for the Laguerre polynomials and associated functions we obtain

$$\Delta_\nu^{(\alpha)}(z,\zeta) = \{E^{(\alpha)}(z,\zeta) + \mathcal{E}_\nu^{(\alpha)}(z,\zeta)\}E_\nu(z,\zeta),\ \nu = 1, 2, 3, \dots, \tag{1.18}$$

where $E^{(\alpha)}(z,\zeta)$ and $\{\mathcal{E}_\nu^{(\alpha)}(z,\zeta)$ are complex functions which are holomorphic in the region $(\mathbb{C}\setminus[0,\infty))\times(\mathbb{C}\setminus[0,\infty))$ and, moreover, $\lim_{\nu\to\infty}\mathcal{E}_\nu^{(\alpha)}(z,\zeta) = 0$ uniformly on each compact subset of this region. We note also that $E^{(\alpha)}(\zeta,\zeta) = 1$ and $\mathcal{E}_\nu^{(\alpha)}(\zeta,\zeta) = 0$ for $\zeta \in \mathbb{C}\setminus[0,\infty)$ and $\nu = 1, 2, 3, \dots$.

From the asymptotic formulas [Chapter III,(2.3)] and [Chapter III, (3.1)] it follows that $E^{(\alpha)}(z,\zeta) = \sum_{j=1}^{2} s_j^{(\alpha)}(z)h_j^{(\alpha)}(\zeta)$, where $s_j^{(\alpha)}, h_j^{(\alpha)}, j = 1, 2$, are holomorphic functions in the region $\mathbb{C}\setminus[0,\infty)$. Then, from (1.16) and (1.18), we obtain that $D_\nu^{(\alpha}(\rho;z)) = \sum_{j=1}^{3} D_{\nu,j}^{(\alpha)}(\rho;z)$, where

$$D_{\nu,j}^{(\alpha)}(\rho;z) = \frac{s_j^{(\alpha)}(z)}{2\pi i}\int_{\Gamma(\lambda,\rho}\frac{h_j^{(\alpha)}(\zeta) - h_j^{(\alpha)}(z)}{\zeta - z}E_\nu(z,\zeta)F(\zeta)\,d\zeta,\ j = 1, 2,$$

and

$$D_{\nu,3}^{(\alpha)}(\rho;z) = \frac{1}{2\pi i}\int_{\Gamma(\lambda,\rho)}\frac{\mathcal{E}_\nu^{(\alpha)}(z,\zeta)}{\zeta - z}E_\nu(z,\zeta)F(\zeta)\,d\zeta.$$

Using Schwarz's Lemma, we prove easily that if $\rho > \max(1, 2\lambda^2, m(\tilde{p}))$ is fixed, then

$$\lim_{\nu\to\infty} D_{\nu,3}^{(\alpha)}(\rho;z) = \lim_{\nu\to\infty}\frac{1}{2\pi i}\int_{\Gamma(\lambda,\rho)}\frac{\mathcal{E}_\nu^{(\alpha)}(z,\zeta)}{\zeta - z}E_\nu(z,\zeta)F(\zeta)\,d\zeta = 0$$

uniformly for $z \in \tilde{p}(\lambda)$.

Let $0 < \delta < \lambda^2$ and denote $\gamma(z,\delta) = p(\lambda)\cap\overline{U}(z;\delta)$. Then

$$D_{\nu,j}^{(\alpha)}(\rho;z) = \frac{s_j^{(\alpha)}(z)}{2\pi i}\left(\int_{\Gamma(\lambda,\rho)\setminus\gamma(z,\delta)} + \int_{\gamma(z,\delta)}\right)H_{\nu,j}^{(\alpha)}(z,\zeta)\,d\zeta,\ j = 1, 2,$$

where $H_{\nu,j}^{(\alpha)}(z,\zeta) = \{h_j^{(\alpha)}(\zeta) - h_j^{(\alpha)}(z)\}(\zeta - z)^{-1}E_\nu(z,\zeta)F(\zeta),\ j = 1, 2.$

Let $z = (\theta - \lambda i)^2, \theta_1 \le \theta \le \theta_2$ be a parametrization of the arc $\tilde{p}(\lambda)(\theta_1$ and θ_2 correspond to the initial and to the endpoint of $\tilde{p}(\lambda)$, respectively), and let

$\zeta = (t+\theta-\lambda i)^2, t_1 \le t \le t_2 (t_1$ and t_2 correspond to the initial and to the end point of $\Gamma(\lambda,\rho)$, respectively, and, evidently, they depend on ρ and z). If we denote $h(t) = F((t-\lambda i)^2)(t-\lambda i)$, then $(j=1,2)$

$$D^{(\alpha)}_{\nu,j}(\rho;z)$$

$$= \frac{s^{(\alpha)}_j(z)}{\pi i}\left(\int_{t_1}^{\tau_1}+\int_{\tau_1}^{\tau_2}+\int_{\tau_2}^{t_2}\right)\frac{h^{(\alpha)}_j(\zeta)-h^{(\alpha)}_j(z)}{t(t+2\theta-2\lambda i)}h(t+\theta)\exp(-2it\sqrt{\nu+1})\,dt,$$

where $t_1 < \tau_1 < 0 < \tau_2 < t_2$, and $\tau_k,\ k=1,2$, correspond to the initial and to the end point of the arc $\gamma(z,\delta) \subset p(\lambda)$, respectively, i.e. they are uniquely determined by the equations $(\tau_k+\theta-\lambda i)^2 = \delta\exp i\varphi_k, 0<\varphi_1<\pi<\varphi_2<2\pi$. Their solutions are $\tau_k = -\delta\sin\varphi_k/(2\lambda), \tau_k\sqrt{(\tau_k+\theta)^2+4\lambda^2} = \mp\delta, k=1,2$, and, hence, $\lim_{\delta\to 0}\tau_k = 0, k=1,2$, uniformly for $\theta\in[\theta_1,\theta_2]$. Moreover,

$$\text{(1.19)}\quad \lim_{\delta\to 0}|\tau_k\delta^{-1}| = \lim_{\delta\to 0}\{(\tau_k+\theta)^2+4\lambda^2\}^{-1/2} = (\theta^2+4\lambda^2)^{-1/2} \ne 0,\infty,\ k=1,2,$$

uniformly for $\theta\in[\theta_1,\theta_2]$.

Since the function $s^{(\alpha)}_j(z)\{h^{(\alpha)}_j(\zeta)-h^{(\alpha)}_j(z)\}(\zeta-z)^{-1},\ j=1,2$, has a removable singularity at the point $\zeta=z$, it is bounded as a function of $\zeta\in\Gamma(\lambda,\rho)$ and $z\in\tilde p(\lambda)$, and this yields the inequalities (j=1,2)

$$\left|\frac{s^{(\alpha)}_j(z)}{\pi i}\int_{\tau_1}^{\tau_2}\frac{h^{(\alpha)}_j(\zeta)-h^{(\alpha)}_j(z)}{t(t+2\theta-2\lambda i)}h(t+\theta)\exp(-2it\sqrt{\nu+1})\,dt\right| \le K\int_{\tau_1}^{\tau_2}|h(t+\theta)|\,dt,$$

where K is a constant not depending on ν and θ.

From (1.19) and the absolutely continuity of the Lebesgue integral we may assert that if $\varepsilon>0$, then there exists $\delta=\delta(\varepsilon)>0$ such that the inequality $K\int_{\tau_1}^{\tau_2}|h(t+\theta)|\,dt < \varepsilon/2$ holds for $\theta\in[\theta_1,\theta_2]$.

Due to the choice of δ, for every $\theta\in[\theta_1,\ \theta_2]$ the function

$$g^{(\alpha)}_j(t,\theta) = \frac{h^{(\alpha)}_j((t+\theta-\lambda i)^2)-h^{(\alpha)}_j((\theta-\lambda i)^2)}{t(t+2\theta-\lambda i)},\ j=1,2,$$

has a bounded variation as a function of $t\in[t_1,\tau_1](j=1)$ and $t\in[\tau_2,t_2](j=2)$, respectively. Moreover, it is bounded on the sets $[t_1,\tau_1]\times[\theta_1,\theta_2](j=1)$ and $[\tau_2,t_2]\times[\theta_1,\theta_2](j=2)$ respectively. Since $h(t)$ is L-integrable on every finite interval and each of the functions $s^{(\alpha)}_j, j=1,2$, is bounded on $\tilde p(\lambda)$, from **(VII.1.1)** it follows that there exists a positive integer ν_0 such that for $\nu>\nu_0$ and $\theta\in[\theta_1,\theta_2]$

$$\left|\frac{s^{(\alpha)}_j(z)}{\pi i}\left(\int_{t_1}^{\tau_1}+\int_{\tau_2}^{t_2}\right)\frac{h^{\alpha}_j(\zeta)-h^{(\alpha)}_j(z)}{t(t+2\theta-2\lambda i)}h(t+\theta)\exp(-2it\sqrt{\nu+1})\,dt\right| < \epsilon/2.$$

This considerations imply $|D^{(\alpha)}_{\nu,j}(\rho;z)| < \varepsilon,\ j = 1,2,$ for $z \in \tilde{p}(\lambda)$ and $\nu > \nu_0$. Thus, the validity of (1.17) is proved.

(VII.1.6) *If* $0 < \lambda < \infty$, *then*

$$\lim_{\rho\to\infty,\nu\to\infty} \frac{1}{2\pi i}\int_{\Gamma(\lambda,\rho)} \frac{1-E_\nu(z,\zeta)}{\zeta - z}\,d\zeta = 1 \tag{1.20}$$

uniformly on every finite arc of $p(\lambda)$.

Proof. Suppose that $\tilde{p}(\lambda)$ is a finite arc of $p(\lambda)$ and let it be parametrized as in the proof of **(VII.1.5)**, i.e. $z = (\theta - \lambda i)^2, \theta_1 \le \theta \le \theta_2, -\infty < \theta_1 \le \theta_2 < \infty$. If $\rho > \max(1, 2\lambda^2, m(\tilde{p}))$, then we parametrize the arc $\Gamma(\lambda,\rho)$ again by $\zeta = (t+\theta -\lambda i)^2, t_1 \le t \le t_2$. It is easy to see that

$$t_1 = -\theta - \sqrt{\rho - \lambda^2},\ t_2 = -\theta + \sqrt{\rho-\lambda^2}. \tag{1.21}$$

If

$$R_\nu(\rho;z) = \frac{1}{2\pi i}\int_{\Gamma(\lambda,\rho)} \frac{1-E_\nu(z,\zeta)}{\zeta - z}\,d\zeta,$$

then

$$R_\nu(\rho;z) = \frac{1}{\pi i}\int_{t_1}^{t_2} \frac{(1-\exp(-2it\sqrt{\nu+1}))(t+\theta-\lambda i)}{t(t+2\theta-2\lambda i)}\,dt \tag{1.22}$$

$$= \frac{1}{2\pi i}\int_{t_1}^{t_2} \frac{1-\exp(-2it\sqrt{\nu+1})}{t}\,dt + \frac{1}{2\pi i}\int_{t_1}^{t_2} \frac{dt}{t+2\theta_2\lambda i}$$

$$-\frac{1}{2\pi i}\int_{t_1}^{t_2} \frac{\exp(-2it\sqrt{\nu+1})}{t+2\theta-2\lambda i}\,dt = \frac{1}{\pi i}\int_{t_1}^{t_2} \frac{\sin^2(2t\sqrt{\nu+1})}{t}\,dt$$

$$+\frac{1}{2\pi}\int_{t_1}^{t_2} \frac{\sin(2t\sqrt{\nu+1})}{t}\,dt + \frac{1}{2\pi i}\int_{t_1}^{t_2} \frac{dt}{t+2\theta-2\lambda i}$$

$$-\frac{1}{2\pi i}\int_{t_1}^{t_2} \frac{\exp(-2it\sqrt{\nu+1})}{t+2\theta-2\lambda i}\,dt.$$

Since the function $t^{-1}\sin^2(2t\sqrt{\nu+1})$ is odd, the inequality

$$\left|\int_{t_1}^{t_2} \frac{\sin^2(2t\sqrt{\nu+1})}{t}\,dt\right| = \left|\int_{-t_1}^{t_2} \frac{\sin^2(2t\sqrt{\nu+1})}{t}\,dt\right|$$

$$\le |\log((-t_1)/t_2)| = \left|\log\frac{\sqrt{\rho-\lambda^2}+\theta}{\sqrt{\rho-\lambda^2}-\theta}\right|$$

holds for $\nu = 0, 1, 2, \ldots, \theta \in [\theta_1, \theta_2]$, hence,

$$\lim_{\rho\to\infty,\nu\to\infty} \frac{1}{\pi i} \int_{t_1}^{t_2} \frac{\sin^2(2t\sqrt{\nu+1})}{t}\, dt = 0 \tag{1.23}$$

uniformly for $\theta \in [\theta_1, \theta_2]$.

Since $\lim_{\rho\to\infty}(-t_1) = \lim_{\rho\to\infty} t_2 = \infty$ when θ runs on $[\theta_1, \theta_2]$, from the equality

$$\frac{1}{2\pi} \int_{t_1}^{t_2} \frac{\sin(2t\sqrt{\nu+1})}{t}\, dt = \frac{1}{2\pi}\left(\int_0^{-t_1\sqrt{\nu+1}} + \int_0^{t_2\sqrt{\nu+1}}\right) \frac{\sin 2t}{t}\, dt$$

it follows

$$\lim_{\rho\to\infty,\nu\to\infty} \int_{t_1}^{t_2} \frac{\sin(2t\sqrt{\nu+1})}{t}\, dt = \frac{1}{\pi}\int_0^{\infty} \frac{\sin 2t}{t}\, dt = \frac{1}{2} \tag{1.24}$$

uniformly for $\theta \in [\theta_1, \theta_2]$.

Having in mind (1.21), we find that

$$\frac{1}{2\pi i}\int_{t_1}^{t_2} \frac{dt}{t+2\theta-2\lambda i} = \frac{1}{2\pi i}\int_{t_1}^{t_2} \frac{t+2\theta}{(t+2\theta)^2+4\lambda^2}\, dt + \frac{\lambda}{\pi}\int_{t_1}^{t_2} \frac{dt}{(t+2\theta)^2+4\lambda^2}$$

$$= \frac{1}{4\pi i}\log\frac{(\theta+\sqrt{\rho-\lambda^2})^2+4\lambda^2}{(\theta-\sqrt{\rho-\lambda^2})^2+4\lambda^2} + \frac{1}{2\pi}\left\{\arctan\frac{\sqrt{\rho-\lambda^2}+\theta}{2\lambda} + \arctan\frac{\sqrt{\rho-\lambda^2}-\theta}{2\lambda}\right\}$$

and, hence,

$$\lim_{\rho\to\infty} \int_{t_1}^{t_2} \frac{dt}{t+2\theta-2\lambda i} = \frac{1}{2} \tag{1.25}$$

uniformly for $\theta \in [\theta_1, \theta_2]$.

Integrating by parts we obtain

$$\int_{t_1}^{t_2} \frac{\exp(-2it\sqrt{\nu+1})}{t+\theta-2\lambda i}\, dt = -\frac{1}{2i\sqrt{\nu+1}}\left\{\frac{\exp(-2it_2\sqrt{\nu+1})}{t_2+2\theta-2\lambda i}\right.$$

$$\left. - \frac{\exp(-2it_1\sqrt{\nu+1})}{t_1+2\theta-2\lambda i} + \int_{t_1}^{t_2} \frac{\exp(-2it\sqrt{\nu+1})}{(t+2\theta-2\lambda i)^2}\, dt\right\}.$$

Hence, for $\theta \in [\theta_1, \theta_2]$ and $\nu = 1, 2, 3, \ldots$ we obtain the inequality

$$\left|\int_{t_1}^{t_2} \frac{\exp(-2it\sqrt{\nu+1})}{t+2\theta-2\lambda i}\, dt\right| \le \frac{1}{2\sqrt{\nu+1}}\left\{\frac{1}{\lambda} + \int_{-\infty}^{\infty} \frac{dt}{t^2+4\lambda^2}\right\},$$

which yields that

$$\lim_{\rho\to\infty,\nu\to\infty} \int_{t_1}^{t_2} \frac{\exp(-2it\sqrt{\nu+1})}{t+2\theta-2\lambda i}\, dt = 0 \tag{1.26}$$

uniformly for $\theta \in [\theta_1, \theta_2]$.

The desired assertion follows from the equalities (1.22), (1.23), (1.24), (1.25) and (1.26).

(VII.1.7) *Suppose that* $0 < \lambda < \infty, \alpha \in \mathbb{R} \setminus \mathbb{Z}^-$, *and that* F *is an* L*-integrable complex function on* $p(\lambda)$ *which satisfies the condition* [Chapter V, (3.28)].

Suppose that F *is bounded on the finite arc* $\tilde{p}(\lambda) \subset p(\lambda)$ *and, moreover, that for each* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon)$ *such that the inequality*

$$\int_{\gamma(z,\delta)} \left| \frac{F(\zeta) - F(z)}{\zeta - z} \right| dm(\lambda) < \varepsilon \tag{1.27}$$

holds for $z \in \tilde{p}(\lambda)$, *where* $m(\lambda)$ *is the Lebesgue measure on* $p(\lambda)$.

Then the series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ *which represents the function* [V, (3.29)] *in the region* $\Delta(\lambda)$ *is uniformly convergent on the arc* $\tilde{p}(\lambda)$ *and*

$$\frac{1}{2\pi i} \int_{p(\lambda)} \frac{F(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2} F(z) \tag{1.28}$$

is its sum.

Proof. From (1.12) it follows that if $z \in \tilde{p}(\lambda), \rho > \max(1, 2\lambda^2, m(\tilde{p}))$ and $\nu = 0, 1, 2, \ldots$, then

$$S_\nu^{(\alpha)}(z) = \frac{1}{2\pi i} \int_{\Gamma(\lambda,\rho)} \frac{1 - E_\nu(z,\zeta)}{\zeta - z} \{F(\zeta) - F(z)\}\, d\zeta \tag{1.29}$$

$$+ \frac{F(z)}{2\pi i} \int_{\Gamma(\lambda,\rho)} \frac{1 - E_\nu(z,\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_{p(\lambda)\setminus\Gamma(\lambda,\rho)} \frac{1 - E_\nu(z,\zeta)}{\zeta - z} F(\zeta)\, d\zeta$$

$$+ r_\nu^{(\alpha)}(z) = \sum_{j=1}^{3} A_{\nu,j}(\rho; z) + r_\nu^{(\alpha)}(z).$$

Since $|E_\nu(z,\zeta)| = 1$ for $z, \zeta \in p(\lambda)$ and $\nu = 0, 1, 2, \ldots$, from [Chapter V, (3.28)] it follows that if $\varepsilon > 0$, then there exists $\rho_0 > \max(1, 2\lambda^2, m(\tilde{p}))$ such that the inequality

$$|A_{\nu,3}(\rho; z)| < \varepsilon/6 \tag{1.30}$$

holds for $z \in \tilde{p}(\lambda), \rho > \rho_0$ and $\nu = 0, 1, 2, \ldots$.

Since the function F is bounded on the arc $\tilde{p}(\lambda)$, due to **(VII.1.6)** we can assert that ρ_0 and the positive integer ν_0 can be chosen in such a way that the inequality

$$|A_{\nu,2}(\rho; z) - F(z)| < \varepsilon/6 \tag{1.31}$$

to hold for $z \in \tilde{p}(\lambda), \rho > \rho_0$, and $\nu > \nu_0$.

The condition (1.27) implies that for $z \in \tilde{p}(\lambda)$ the integral

$$\frac{1}{2\pi i}\int_{p(\lambda)} \frac{F(\zeta)}{\zeta - z}\,d\zeta$$

exists as a main value in the Cauchy sence. If $\zeta_j,\ j = 1, 2$, are the end points of the arc $\Gamma(\lambda, \rho)$, then for $z \in \tilde{p}(\lambda)$ it holds the equality

$$A_{\nu,1}(\rho; z) = \frac{1}{2\pi i}\int_{\Gamma(\lambda,\rho)} \frac{F(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2}F(z) - \frac{F(z)}{2\pi i}\log\frac{\zeta_1 - z}{\zeta_2 - z} \tag{1.32}$$

$$-\frac{1}{2\pi i}\int_{\Gamma(\lambda,\rho)} \frac{F(\zeta) - F(z)}{\zeta - z}E_\nu(z,\zeta)\,d\zeta$$

Since $\lim_{\rho\to\infty}\log\{(\zeta_1 - z)/(\zeta_2 - z)\} = 0$ uniformly for $z \in \tilde{p}(\lambda)$ and F is bounded on the arc $\tilde{p}(\lambda)$, the inequality

$$|(2\pi i)^{-1}F(z)\log\{(\zeta_1 - z)/(\zeta_2 - z)\}| < \varepsilon/6$$

holds for $z \in \tilde{p}(\lambda)$ provided that $\rho > \rho_0$ and ρ_0 is large enough.

It remains to study the integral

$$\frac{1}{2\pi i}\int_{\Gamma(\lambda,\rho)} \frac{F(\zeta) - F(z)}{\zeta - z}E_\nu(z,\zeta)\,d\zeta$$

$$= \frac{1}{2\pi i}\left(\int_{\Gamma(\lambda,\rho)\setminus\gamma(z,\delta)} + \int_{\gamma(z,\delta)}\right)\frac{F(\zeta) - F(z)}{\zeta - z}E_\nu(z,\zeta)\,d\zeta$$

when $\delta > 0$ is small enough.

There exits a constant $M > 0$ not depending on z and ν, and such that for $z \in \tilde{p}(\lambda)$ and $\nu = 0, 1, 2, \ldots,$

$$\left|\frac{1}{2\pi i}\int_{\gamma(z,\delta)} \frac{F(\zeta) - F(z)}{\zeta - z}E_\nu(z,\zeta)\,d\zeta\right| \le M\int_{\gamma(z,\delta)}\left|\frac{F(\zeta) - F(z)}{\zeta - z}\right|dm(\lambda).$$

From the above inequality as well as from the condition (1.27) it follows that there exists $\delta = \delta(\epsilon) > 0$ such that for $z \in \tilde{p}(\lambda)$ and $\nu = 0, 1, 2, \ldots$

$$\left|\frac{1}{2\pi i}\int_{\gamma(z,\delta)} \frac{F(\zeta) - F(z)}{\zeta - z}E_\nu(z,\zeta)\,d\zeta\right| < \varepsilon/6.$$

If this δ is fixed, then by means of **(VII.1.1)** we prove the inequality

$$\left|\frac{1}{2\pi i}\int_{\Gamma(\lambda,\rho)\setminus\gamma(z,\delta)} \frac{F(\zeta) - F(z)}{\zeta - z}E_\nu(z\zeta)\,d\zeta\right| < \varepsilon/6$$

for $z \in \tilde{p}(\lambda)$ provided ν is large enough.

Let $\rho > \rho_0$ be chosen such that for $z \in \tilde{p}(\lambda)$

$$\left| \frac{1}{2\pi i} \int_{p(\lambda)} \frac{F(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \frac{F(\zeta)}{\zeta - z}\, d\zeta \right| < \varepsilon/6.$$

Then, from (1.32) it follows that for $z \in \tilde{p}(\lambda)$ and for sufficiently large ν

$$\left| A_{\nu,1}(\rho; z) - \frac{1}{2\pi i} \int_{p(\lambda)} \frac{F(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2} F(z) \right| < \varepsilon/2 \tag{1.33}$$

Further, **(VII.1.5)** yields that $|r_\nu^{(\alpha)}(z)| < \varepsilon/6$ for $z \in \tilde{p}(\lambda)$ provided ν is large enough. Then, taking into account (1.29), (1.30), (1.31) and (1.33), we conclude that for such z and ν

$$\left| S_\nu^{(\alpha)}(z) - \frac{1}{2\pi i} \frac{F(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2} F(z) \right| < \varepsilon.$$

Remark. If the function F satisfies a Hölder condition on the arc $\tilde{p}(\lambda)$, i.e. if there exist constants $K > 0$ and $0 < \kappa \le 1$ such that $|F(\zeta) - F(z)| \le |\zeta - z|^\kappa$ for $\zeta, z \in \tilde{p}(\lambda)$, then the requirement (1.27) is satisfied.

As a corollary of **(VII.1.7)** we can state the following proposition:

(VII.1.8) *Suppose that* $0 < \lambda < \infty, \alpha \in \mathbb{R} \setminus \mathbb{Z}^-$, *and let* f *be a complex function which is continuous on the closed region* $\overline{\Delta}(\lambda)$ *and holomorphic in* $\Delta(\lambda)$. *Suppose that* f *satisfies the following conditions*:

(a) *there exists* $\delta > 0$ *such that* $|f(z)| = O(|z|^{1/2-\delta})$ *when* $z \to \infty$ *in* $\Delta(\lambda)$;

(b) $\displaystyle\int_{p(\lambda)} |z|^{\sigma(\alpha)} |f(z)| dm(\lambda) < \infty,\ \sigma = \max(-1, \alpha/2 - 5/4).$

If f *satisfies Hölder condition on the finite arc* $\tilde{p}(\lambda) \subset p(\lambda)$, *then the series in the Laguerre polynomials* $\{L_n^{(\alpha)}(z)\}_{n=0}^\infty$ *which represents* f *in the region* $\Delta(\lambda)$ *is uniformly convergent on* $\tilde{p}(\lambda)$ *and* $f(z)$ *is its sum for each* $z \in \tilde{p}(\lambda)$.

2. (C,δ)-summability on the boundaries of convergence regions

2.1 Let us remind that a series

$$\sum_{n=0}^{\infty} u_n,\ u_n \in \mathbb{C},\ n = 0, 1, 2, \dots \tag{2.1}$$

is said to be summable by the Cesaro means or, breafly, Cesaro-summable, if there exists

$$\sigma = \lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \frac{1}{n+1} \{s_0 + s_1 + s_2 + \cdots + s_n\},$$

where $s_\nu = \sum_{k=0}^{\nu} u_k,\ \nu = 0, 1, 2, \ldots.$

Usually the complex number σ is called Cesaro sum of the series (2.1). Every convergent series is Cesaro-summable and, moreover, its Cesaro sum is equal to its sum in the usual sence. In general, the converse is not true.

The formal identity

$$(1-z)^{-2}\sum_{n=0}^{\infty} u_n z^n = \sum_{n=0}^{\infty}(n+1)\sigma_n z^n, \tag{2.2}$$

as well as the expansion

$$(1-z)^{-2} = \sum_{n=0}^{\infty}(n+1)z^n,\ |z|<1, \tag{2.3}$$

provide a generalization of the Cesaro summability which is based on the fact that the Cesaro means $\{\sigma_n\}_{n=0}^{\infty}$ can be obtained by dividing the coefficients of the series on the right-hand side of (2.2) by those of the series in (2.3).

We define the sequence $\{\tilde{\sigma}_n^{(\delta)}\}_{n=0}^{\infty},\ \delta \in \mathbb{C}\setminus\mathbb{Z}^-$, by the (formal) expansion

$$(1-z)^{-1-\delta}\sum_{n=0}^{\infty} u_n z^n = \sum_{n=0}^{\infty} \tilde{\sigma}_n^{(\delta)} z^n.$$

Then, having in view the series representation

$$(1-z)^{-1-\delta} = \sum_{n=0}^{\infty}\binom{n+\delta}{n} z^n,\ |z|<1,$$

we say that the series (2.1) is (C,δ)-summable if there exists

$$\sigma^{(\delta)} = \lim_{n\to\infty}\sigma_n^{(\delta)} = \lim_{n\to\infty}\binom{n+\delta}{n}^{-1}\tilde{\sigma}_n^{(\delta)},$$

and then we use the notation

$$(C,\delta)\sum_{n=0}^{\infty} u_n = \sigma^{(\delta)}.$$

From the formal expansion

$$(1-z)^{-1}\sum_{n=0}^{\infty} u_n z^n = \sum_{n=0}^{\infty} s_n z^n$$

it follows that $(C,0)$-summability is, in fact, the usual convergence and $(C,1)$-summability is nothing but the Cesaro summability.

It is well-known that if the series (2.1) is (C,δ)-summable for some $\delta > -1$, then it is $(C,\tilde{\delta})$-summable for every $\tilde{\delta} > \delta$ and, moreover,

$$(C,\tilde{\delta})\sum_{n=0}^{\infty} u_n = (C,\delta)\sum_{n=0}^{\infty} u_n.$$

In particular, if a series is convergent with sum s, then it is (C,δ)-summable for every $\delta > 0$ and $\sigma^{(\delta)} = s$.

The proof of the validity of the combinatorial identity

$$\sum_{k=0}^{n}\binom{k+\delta-1}{k} = \binom{n+\delta}{n},\ n = 0,1,2,\ldots; \tag{2.4}$$

$$\sigma_n^{(\delta)} = \sum_{k=0}^{n}\binom{n-k+\delta-1}{n-k} s_k = \sum_{k=0}^{n} A_{n-k}^{(\delta-1)} s_k, \tag{2.5}$$

where

$$A_k^{(\delta)} = \binom{k+\delta}{k} = \frac{\Gamma(k+\delta+1)}{\Gamma(k+1)},\ \delta \in \mathbb{C}\setminus\mathbb{Z}^-,\ k = 0,1,2,\ldots, \tag{2.6}$$

is left to the reader as easy exercises.

2.2 Now we are going to study the (C,δ)-summability of series in Jacobi polynomials on the boundaries of their regions of convergence. To this end we shall use the notations introduced in the proof of the proposition **(VII.1.2)**. More precisely, if $z_0 = u(\theta_0) = (r\exp i\theta_0 + r^{-1}\exp i\theta_0)/2, -\pi < \theta_0 \le \pi$, is a point on the ellipse $e(r)$, then we define the function

$$\tilde{\Phi}(t) = \int_0^t |\tilde{\varphi}(\theta+\theta_0) - \tilde{\varphi}(\theta_0)|\,d\theta \tag{2.7}$$

(VII.2.1) *Let $\alpha+1, \beta+1$ and $\alpha+\beta+2$ be not equal to $0,-1,-2,\ldots,$ and let $1 < r < \infty$. Suppose that for some point $z_0 \in e(r)$*

$$\tilde{\Phi}(t) = o(t),\ |t| \to 0, \tag{2.8}$$

and, moreover, that the integral in (1.9) *exists as a Cauchy main value for $z = z_0$. Then the series in Jacobi polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ representing the function* [Chapter V, (1.1)] *in the region $E(r)$, is (C,δ)-summable at the point z_0 for each $\delta > 0$ with sum* (1.9), *where $z = z_0$.*

Proof. Suppose that $0 < \delta < 1$ and let

$$\sigma_n^{(\delta)}(\alpha,\beta;z) = \binom{n+\delta}{n}^{-1} \sum_{k=0}^{n} \binom{n-k+\delta-1}{n-k} S_k^{(\alpha,\beta}(z)$$

be the corresponding Cesaro means of order δ. From the asymptotic formula (1.3) we obtain

$$\sigma_n^{(\delta)}(\alpha,\beta;z) = \frac{1}{2\pi i}\int_{e(r)} \frac{1-K_n^{(\delta)}(z,\zeta)}{\zeta - z}\varphi(\zeta)\,d\zeta + R_n^{(\delta)}(\alpha,\beta;z), \tag{2.9}$$

for $z \in e(r)$, where

$$K_n^{(\delta)}(z,\zeta) = \binom{n+\delta}{n}^{-1} \sum_{k=0}^{n} \binom{n-k+\delta-1}{n-k} (\omega(z)/\omega(\zeta))^k \tag{2.10}$$

and

$$R_n^{(\delta)}(\alpha,\beta;z) = \binom{n+\delta}{n}^{-1} \sum_{k=0}^{n} \binom{n-k+\delta-1}{n-k} r_k^{(\alpha,\beta)}(z).$$

Moreover, since $\lim_{n\to\infty} r_n^{(\alpha,\beta)}(z) = 0$ for each $z \in e(r)$, it is easy to prove that $\lim_{n\to\infty} R_n^{(\delta)}(\alpha,\beta;z) = 0$ for $z \in e(r)$.

Suppose that n_0 is a positive integer such that $n_0^{-1} < d(r)$ =dist$(e(r),[-1,1])$ and define $e_n(r) = e(r) \setminus e(z_0, 1/n)$, and

$$T_n^{(\delta)}(\alpha,\beta;z_0) = \sigma_n^{(\delta)}(\alpha,\beta;z_0) - \varphi(z_0) - \frac{1}{2\pi i}\int_{e_n(z_0)} \frac{\varphi(\zeta)-\varphi(z_0)}{\zeta - z_0}\,d\zeta \tag{2.11}$$

for $n > n_0$.

From (1.11) and (2.4) it follows that $(n > n_0)$

$$\sigma_n^{(\delta)}(\alpha,\beta;z_0) - \varphi(z_0) = \frac{1}{2\pi i}\int_{e(r)} \frac{1-K_n^{(\delta)}(z_0,\zeta)}{\zeta - z)_0}\{\varphi(\zeta)-\varphi(z_0)\}\,d\zeta + R_n^{(\delta)}(\alpha,\beta;z_0).$$

Having in mind (2.10), we find that

$$T_n^{(\delta)}(\alpha,\beta;z_0) = -\frac{1}{2\pi i}\int_{e_n}(z_0) \frac{K_n^{(\delta)}(z_0,\zeta)}{\zeta - z_0}\{\varphi(\zeta)-\varphi(z_0)\}\,d\zeta \tag{2.12}$$

$$+\frac{1}{2\pi i}\int_{e(z_0,1/n)} \frac{1-K_n^{(\delta)}(z_0,\zeta)}{\zeta - z_0}\{\varphi(\zeta)-\varphi(z_0)\}\,d\zeta + R_n^{(\delta)}(\alpha,\beta;z_0)$$

$$= P_n^{(\delta)}(z_0) + Q_n^{(\delta)}(z_0) + R_n^{(\delta)}(\alpha,\beta;z_0).$$

Suppose that $z_0 = u(\theta_0) \neq (r + r^{-1})/2$ and denote by $z_{j,n}, j = 1, 2$, the endpoints of the arc $e(z_0, 1/n)$. Then $z_{j,n} = u(\theta_{j,n}), j = 1, 2$, and $-\pi < \theta_{1,n} < \theta_0 < \theta_{2,n} < \pi$. If $z_0 = -(r + r^{-1})/2$, i.e. $\theta_0 = \pi$, then we choose $\theta_{j,n}, j = 1, 2$, such that $0 < \theta_{1,n} < \pi < \theta_{2,n} < 2\pi$. As it is easily seen, in both cases $\theta_{j,n} - \theta_0 = O(n^{-1})$ when $n \to \infty$, $j = 1, 2$.

We suppose again that $z_0 \neq (r + r^{-1})/2$ but our considerations hold also for $z_0 = (r + r^{-1})/2$. By means of the parametrization $\zeta = u(t), -\pi \leq t \leq \pi$, we obtain that $P_n^{(\delta)}(z_0) = \sum_{k=1}^{2} P_{n,k}^{(\delta)}(z_0)$, where

$$P_{n,1}^{(\delta)}(z_0) = \int_{-\pi}^{\theta_{1,n}} p_n^{(\delta)}(\theta_0, t)\, dt, \quad P_{n,2}^{(\delta)}(z_0) = \int_{\theta_{2,n}}^{\pi} p_n^{(\delta)}(\theta_0, t)\, dt$$

and

$$-4\pi p_n^{(\delta)}(\theta_0, t) \tag{2.13}$$

$$= \frac{K_n^{(\delta)}(u(\theta_0), u(t))}{u(t) - u(\theta_0)} \{\tilde{\varphi}(t) - \tilde{\varphi(\theta_0)}\}(r \exp it - r^{-1} \exp(-it)), \ -\pi \leq t \leq \pi.$$

Assume that $0 \leq \theta_0 < \pi$. Then, setting $t = \theta + \theta_0$ in the integral defining $P_{n,2}^{(\delta)}(z_0)$, we obtain that $P_{n,2}^{(\delta)}(z_0) = \int_{t_{2,n}}^{\pi-\theta_0} p_n^{(\delta)}(\theta_0, \theta+\theta_0)\, d\theta$, where $0 < t_{2,n} < \pi - \theta_0$ and $t_{2,n} = \theta_{2,n} - \theta_0 = O(n^{-1})$ when $n \to \infty$. Then, having in mind (2.13), we can assert that there exists a positive constant A such that

$$|P_{n,2}^{\delta}(z_0)| \leq A \int_{t_{2,n}}^{\pi-\theta_0} \frac{|K_n^{(\delta)}(u(\theta_0), u(\theta + \theta_0))|}{\sin(\theta/2)} |\tilde{\varphi}(\theta + \theta_0) - \tilde{\varphi}(\theta_0)|\, d\theta. \tag{2.14}$$

Since $0 < \delta < 1$, the power series expansion of the function $(1 - w)^{-\delta}$ in the disk $U(0; 1)$ converges for each $w \neq 1$ with $|w| = 1$, and as a corollary of (2.10) we obtain that if $0 < \theta < \pi$, then

$$\binom{n+\delta}{n} |K_n^{(\delta)}(u(\theta_0), u(\theta + \theta_0))| = \left| \sum_{k=0}^{n} \binom{k+\delta-1}{k} \exp ik\theta \right| \tag{2.15}$$

$$= \left| (1 - \exp i\theta)^{-\delta} - \sum_{k=n+1}^{\infty} \binom{k+\delta-1}{k} \exp ik\theta \right|.$$

Since $0 < \delta < 1$, the sequence $\left\{ \binom{k+\delta-1}{k} \right\}_{k=0}^{\infty}$ decreases and tends to zero. Hence, for $0 < \theta < \pi$ we have the estimate

$$\left| \sum_{k=n+1}^{\infty} \binom{k+\delta-1}{k} \exp ik\theta \right| \leq 2 \binom{n+\delta}{n+1} |1 - \exp i\theta|^{-1}. \tag{2.16}$$

Using (2.15) and (2.16), we obtain that $(0<\theta<\pi)$

$$\frac{|K_n^{(\delta)}(u(\theta_0),u(\theta+\theta_0))|}{\sin(\theta/2)} \le \frac{1}{\binom{n+\delta}{n}\sin(\theta/2)}\left\{\frac{1}{(2\sin(\theta/2))^{\delta}}+\frac{\binom{n+\delta}{n+1}}{\sin(\theta/2)}\right\}.$$

Further, from the above inequality, the inequality $\sin(\theta/2)\ge\theta/\pi, 0<\theta<\pi$, as well as from the asymptotic formula

$$\binom{n+\delta}{n}=\frac{n^{\delta}}{\Gamma(1+\delta)}\{1+O(n^{-1})\},\ n\to\infty,$$

it follows the existence of a constant B not depending on n and θ such that $|K_n^{(\delta)}(u(\theta_0),u(\theta+\theta_0))|\le B\theta^{-1-\delta}n^{-\delta}$ for $0<\theta<\pi$, and when n is large enough.

Then (2.14) and (2.7) yield that

$$|P_{n,2}^{(\delta)}(z_0)|\le ABn^{-\delta}\int_{t_{2,n}}^{\pi-\theta_0}|\tilde\varphi(\theta+\theta_0)-\tilde\varphi(\theta_0)|\theta^{-1-\delta}\,d\theta$$

$$=ABn^{-\delta}\int_{t_{2,n}}^{\pi-\theta_0}\theta^{-1-\delta}\tilde\Phi'(\theta)\,d\theta=ABn^{-\delta}\Big\{\tilde\Phi(\pi-\theta_0)(\pi-\theta_0)^{-1-\delta}$$

$$-\tilde\Phi(t_{2,n})t_{2,n}^{-1-\delta}+AB(1+\delta)\int_{t_{2,n}}^{\pi-\theta_0}\tilde\Phi(\theta)\theta^{-2-\delta}\,d\theta\Big\}.$$

Using the above estimate along with the condition (2.8), we obtain that $\lim_{n\to\infty}P_{n,2}^{(\delta)}(z_0)=0$. In the same way we prove that $\lim_{n\to\infty}P_{n,1}^{(\delta)}(z_0)=0$ and, hence, $\lim_{n\to\infty}P_n^{(\delta)}(z_0)=0$.

It is easily seen that there exists a constant C, not depending on δ and n such that

$$|Q_n^{(\delta)}(z_0)|\le C\int_{t_{1,n}}^{t_{2,n}}\frac{|1-K_n^{(\delta)}(u(\theta_0),u(\theta+\theta_0))|}{|\sin(\theta/2)|}|\tilde\varphi(\theta+\theta_0)-\tilde\varphi(\theta_0)|\,d\theta.$$

Since $|\sin k\theta|\le k|\sin\theta|,\ k=0,1,2,\dots$, the equality (2.5) yields

$$\frac{|1-K_n^{(\delta)}(u(\theta_0),u(\theta+\theta_0))}{|\sin(\theta/2)|}=\frac{\left|\sum_{k=0}^{n}\binom{n-k+\delta-1}{n-k}(1-\exp(-ik\theta))\right|}{\binom{n+\delta}{n}|\sin(\theta/2)|}$$

$$\le\frac{2}{\binom{n+\delta}{n}}\sum_{k=0}^{n}\binom{n-k+\delta-1}{n-k}\frac{|\sin(k\theta/2)|}{|\sin(\theta/2)|}$$

$$\leq \frac{2n}{\binom{n+\delta}{n}} \sum_{k=0}^{n} \binom{n-k+\delta-1}{n-k} = 2n.$$

Hence,

$$|Q_n^{(\delta)}(z_0)| \leq 2Cn \int_{t_{1,n}}^{t_{2,n}} |\tilde{\varphi}(\theta+\theta_0) - \varphi(\theta_0)|\, d\theta = 2Cn\{\tilde{\Phi}(t_{2,n}) - \Phi(\tilde{t}_{1,n})\},$$

and then from (2.6) it follows that $\lim_{n\to\infty} Q_n^{(\delta)}(z_0) = 0$.

So far we have proved that

$$\lim_{n\to\infty} \left\{ \sigma_n^{(\delta)}(\alpha,\beta;z_0) - \varphi(z_0) - \frac{1}{2\pi i}\int_{e_n(r)} \frac{\varphi(\zeta)-\varphi(z_0)}{\zeta - z_0}\, d\zeta \right\} = 0.$$

Then, from the existence of the integral (1.10) as a Cauchy main value, we have

$$\lim_{n\to\infty} \sigma_n^{(\delta)}(\alpha,\beta;z_0) = \frac{1}{2\pi i}\int_{e(r)} \frac{\varphi(\zeta)}{\zeta - z_0}\, d\zeta + \frac{1}{2}\varphi(z_0).$$

The validity of the last equality was proved under the assumption that $0 < \delta < 1$ but a property of the (C,δ) summation, mentioned above, gives that it holds for every $\delta \geq 1$.

As an immediate corollary of **(VII.2.1)** we can state the following proposition:

(VII.2.2) *Suppose that* $\alpha+1, \beta+1$ *and* $\alpha+\beta+2$ *are not equal to* $0,-1,-2,\dots, 1<r<\infty$, *and let* f *be a complex function which is continuous on* $\overline{E(r)}$ *and holomorphic in* $E(r)$. *If* $\delta > 0$, *then the series in the Jacobi polynomials* $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ *which represents* f *in the region* $E(r)$ *is* (C,δ)*-summable at each point* $z_0 \in e(r)$ *with* (C,δ)*-sum* $f(z_0)$.

2.3 For series in Laguerre, and respectively, Hermite, polynomials it is desirable to have propositions like **(VII.2.2)** but, at present, we shall confine ourselves on classes of functions having Cauchy type integral representations. The main reason is that in order to obtain suitable asymptotic formulas for the corresponding Cesaros means, for the coefficients of the series expansions in the Laguerre or Hermite polynomials of the functions under consideration we need integral representations in terms of the corresponding associated functions.

As usually, we shall begin by proving an auxiliary proposition which plays the role of the "key Lemma". To this end, we define

$$(2.17)\quad W_n^{(\delta)}(t) = \sum_{k=0}^{n} A_{n-k}^{(\delta-1)} \exp(it\sqrt{k+1}),\ t\in\mathbb{R},\ \delta\in\mathbb{C}\setminus\mathbb{Z}^-,\ n=0,1,2,\dots,$$

where $A_k^{(\delta)},\ k=0,1,2,\dots$, are given by the equalities (2.6).

We need the asymptotics of $W_n^{(\delta)}(t)$ when n tends to infinity and $|t|$ is bounded. The following assertion holds true:

(VII.2.3) *If* $0 < \delta < 1$ *and* $q \in (0, \min(1/2, \delta))$, *then there exist functions* $W_{n,j}^{(\delta)}(q;t)$, $j = 1,2,3$, $t \in \mathbb{R}, n = 0,1,2,\dots$, *such that* $W_{n,1}^{(\delta)}(q;t) = O(1), W_{n,2}^{(\delta)}(q;t) = O(n^{\delta-1}t^{-2})$, $W_{n,3}^{(\delta)}(q;t) = O(n^{\delta/2}|t|^{-\delta})$ *uniformly with respect to* t *on each set of the kind* $(-T,0)\cup(0,T)$ *with* $T \in \mathbb{R}^+$ *when* n *tends to infinity and, moreover*

$$W_n^{(\delta)}(t) = \sum_{j=1}^{3} W_{n,j}^{\delta}(q;t),\ t \in \mathbb{R}. \tag{2.18}$$

Proof. Define

$$w_n^{(\delta)}(t;z) = \frac{\Gamma(n-z+\delta)\exp(it\sqrt{z+1})}{\Gamma(\delta)\Gamma(n-z+1)},\ n = 0,1,2,\dots \tag{2.19}$$

for $t \in \mathbb{R}$ and $z \in \mathbb{C}\setminus\{(-\infty,-1]\cup[n+\delta,\infty)\}$. Then, having in mind (2.17), we obtain

$$W_n^{(\delta)}(t) = \sum_{k=0}^{n} w_n^{(\delta)}(t;k).$$

If $\{z_j\}_{j=1}^{s} \subset \mathbb{C}$, then we denote by $[z_1, z_2, z_3, \dots, z_s]$ the polygonal line joining successively the points $z_1, z_2, z_3, \dots, z_s$.

We define $a(z) = \pi(\cot\pi z + i)$ for $z \in \mathbb{C}\setminus\mathbb{Z}$ and $\operatorname{Im} z \ge 0$, $a(z) = \pi(\cot\pi z - i)$ when $\operatorname{Im} z < 0$, and

$$I_n^{(\delta)}(q;t) = \frac{1}{2\pi i}\int_{L_n(q,R)} w_n^{(\delta)}(t;z)a(z)\,dz, \tag{2.20}$$

where $L_n(q,R) = [n+q-iR, n+q+iR, -q+iR, q-iR, n+q-iR]$ provided $R > 0$.

Remark. It is clear that the integral in (2.20) does not depend on R.

If $L_{n,1}(q,R) = [n+q, n+q+iR, -q+iR, -q]$ and $L_{n,2}(q,R) = [-q, -q-iR, n-iR, n+q]$, then the residue theorem as well as the Cauchy integral theorem yield

$$2\pi i I_n^{(\delta)}(q,t)$$

$$= \int_{L_{n,1}(q,R)} w_n^{(\delta)}(t;z)\pi(\cot\pi z + i)\,dz + \int_{L_{n,2}(q,R)} w_n^{(\delta)}(t;z)\pi(\cot\pi z - i)\,dz$$

$$= \int_{L_n(q,R)} w_n^{(\delta)}(t;z)\pi\cot\pi z\,dz + \pi i\int_{L_{n,1}(q,R)} w_n^{(\delta)}(t;z)\,dz - \pi i\int_{L_{n,2}(q,R)} w_n^{(\delta)}(t;z)\,dz$$

$$= 2\pi i\Big\{\sum_{k=0}^{n} w_n^{(\delta)}(t;k) + \int_{-q}^{n+q} w_n^{(\delta)}(t;x)\,dx\Big\}.$$

Hence, $W_n^{(\delta)}(t) = I_n^{(\delta)}(q;t) + H_n^{(\delta)}(q;t)$, where

$$H_n^{\delta}(q;t) = -\int_{-q}^{n+q} w_n^{(\delta)}(t;x)\,dx. \tag{2.21}$$

Further, we have $I_n^{(\delta)}(q;t) = \sum_{j=1}^{6} K_{n,j}(\delta)(q,R;t)$, where

$$K_{n,1}^{(\delta)}(q,R;t) = i\exp(2\lambda\pi i)\int_0^R \frac{w_n^{(\delta)}(t;n+q+iy)\exp(-2\pi y)}{\exp(2q\pi i)\exp(-2\pi y)-1}\,dy,$$

$$K_{n,2}^{(\delta)}(q,R;t) = i\exp(-2q\pi i)\int_0^R \frac{w_n^{(\delta)}(t;n+q-iy)\exp(-2\pi y}{1-\exp(-2q\pi i)\exp(-2\pi y)}\,dy,$$

$$K_{n,3}^{(\delta)}(q,R;t) = -i\exp(2\lambda\pi i)\int_0^R \frac{w_n^{(\delta)}(t;-q+iy)\exp(-2\pi y)}{\exp(-2q\pi i)\exp(-2\pi y)-1}\,dy,$$

$$K_{n,4}^{(\delta)}(q,R;t) = -i\exp(2q\pi i)\int_0^R \frac{w_n^{(\delta)}(t;-q-iy)\exp(-2\pi y)}{1-\exp(2q\pi i)\exp(-2\pi y)}\,dy,$$

$$K_{n,5}^{(\delta)}(q,R;t) = -\exp(-2\pi R)\int_{-q}^{n+q} \frac{w_n^{(\delta)}(t;x+iR)\exp(2\pi ix)}{\exp(-2\pi R)\exp(2\pi ix)-1}\,dx,$$

$$K_{n,6}^{(\delta)}(q,R;t) = \exp(-2\pi R)\int_{-q}^{n+\lambda} \frac{w_n^{(\delta)}(t;x-iR)\exp(-2\pi ix)}{1-\exp(-2\pi R)\exp(-2\pi ix)}\,dx.$$

If δ, q and n are fixed, then the asymptotic formula [H. Bateman, A. Erdélyi, 1, 1.18, (6)] we obtain $|\Gamma(n-x+\delta\pm iR)/\Gamma(n-x\pm iR)| = O(R^{\delta-1})$ uniformly with respect to $x\in(-q,n+q)$ when $R\to\infty$. Then from (2.19) we get $|w_n^{(\delta)}(t;x\pm iR)| = O(R^{\delta-1}\exp(|t|\sqrt{R+n+q+1}))$ uniformly with respect to $x\in(-q,n+q)$ when $t\in\mathbb{R}$ is fixed and $R\to\infty$. Hence, $\lim_{R\to\infty} K_{n,5}^{(\delta)}(q,R;t) = \lim_{R\to\infty} K_{n,6}^{(\delta)}(q,R;t) = 0$ for $t\in\mathbb{R}$. Thus, we obtain the representation $I_n^{(\delta)}(q;t) = \sum_{j=1}^{4} I_{n,j}^{(\delta)}(q;t)$, where

$$I_{n,1}^{(\delta)}(q;t) = i\exp(2q\pi i)\int_0^\infty \frac{w_n^{(\delta)}(t;n+q+iy)\exp(-2\pi y)}{\exp(2q\pi i)\exp(-2\pi y)-1}\,dy,$$

$$I_{n,2}^{(\delta)}(q;t) = i\exp(-2q\pi i)\int_0^\infty \frac{w_n^{(\delta)}(t;n+q-iy)\exp(-2\pi y)}{\exp(2q\pi i)\exp(-2\pi y)-1}\,dy,$$

$$I_{n,3}^{(\delta)}(q;t) = -i\exp(-2q\pi i)\int_0^\infty \frac{w_n^{(\delta)}(r;-q+iy)\exp(-2\pi y)}{\exp(-2q\pi i)\exp(-2\pi y)-1}\,dy,$$

$$I_{n,4}^{(\delta)}(q;t) = -i\exp(2\pi i)\int_0^\infty \frac{w_n^{(\delta)}(t;-q-iy)\exp(-2\pi y)}{1-\exp(2q\pi i)\exp(-2\pi y)}\,dy.$$

Since $|\Gamma(\delta-q\pm iy)/\Gamma(1-q\pm iy)| = O(1), |\exp(\pm 2q\pi i)\exp(-2\pi y)-1| \geq \sin(2q\pi)$ and $|\operatorname{Im}(n\pm q\pm iy)^{1/2}| \leq y^{1/2}$ when $n = 0,1,2,\ldots$ and $0 \leq y < \infty$, we have that $|w_n^{(\delta)}(t;n\pm q\pm iy)| = O(\exp(Ty^{1/2}))$ when $|t| \leq T, 0 \leq y < \infty$, and $n = 0,1,2\ldots.$ Hence, $I_{n,j}^{(\delta)}(q;t) = O(\int_0^\infty \exp(-2\pi y + Ty^{1/2})\,dy) = O(1), j = 1,2$, uniformly with respect to $t \in (-T,T)$ and $n = 0,1,2\ldots.$

The asymptotic formula [H. Bateman, A. Erdélyi, 1, 1.18,(4)] yields

$$|\Gamma(n-q+\delta\pm iy)/\Gamma(n-q+1\pm iy)| = O((n^2+y^2)^{(\delta-1)/2}) = O(n^{\delta-1})$$

uniformly with respect to $y \in (0,\infty)$ when n tends to infinity. Hence,

$$I_{n,j}^{(\delta)}(q;t) = O\Big(n^{\delta-1}\int_0^\infty \exp(-2\pi y + Ty^{1/2})\,dy\Big) = O(n^{\delta-1}) = O(1), j = 3,4,$$

uniformly with respect to $t \in (-T,T)$ for $n \to \infty$.

If we integrate $w_n^{(\delta)}(t;z)$ as a function of the complex variable z along each of the closed polygons $[-q, n+q, n+q+iR, -q+iR, -q]$ and $[-q, -q-iR, n+q-iR, n+q, -q]$, then the Cauchy integral theorem yields

$$(2.22)\qquad \int_{-q}^{n+q} w_n^{(\delta)}(t;x)\,dx + i\int_0^R w_n^{(\delta)}(t;n+q+iy)$$
$$-i\int_0^R w_n^{(\delta)}(t;-q+iy)\,dy - \int_{-q}^{n+q} w_n^{(\delta)}(t;x+iR)\,dx = 0$$

and

$$(2.23)\qquad -\int_{-q}^{n+q} w_n^{(\delta)}(t;x)\,dx - i\int_0^R w_n^{(\delta)}(t;-q-iy)\,dy$$
$$+i\int_0^R w_n^{(\delta)}(t;n+q-iy)\,dy + \int_{-q}^{n+q} w_n^{(\delta)}(t;x-iR)\,dx = 0.$$

Suppose that n is fixed. Then, since $\operatorname{signIm}\{(x+1\pm iR)^{1/2}\} = \pm 1$, $|w_n^{(\delta)}(t;x+iR)| = O(R^{\delta-1})$ if $t > 0$ and $|w_n^{(\delta)}(t;x-iR)| = O(R^{\delta-1})$ if $t < 0$ uniformly with respect to $x \in (-q,n+q)$. Therefore, $\lim_{R\to\infty}\int_{-q}^{n+q} w_n^{(\delta)}(t;x+iR)\,dx = 0$ if $t > 0$ and $\lim_{R\to\infty}\int_{-q}^{n+q} w_n^{(\delta}(t;x-iR))\,dx = 0$ if $t < 0$.

If $h(y) = 2^{-1/2}((1+y^2)^{1/2}-1)^{1/2}, 0 \leq y < \infty$, then, by an easy computation, we get $h(y) \geq (1/2)y^{1/2}$ for $y \geq 4/3$. Moreover, it is clear that $m = \inf_{0<y\leq 4/3} y^{-1}h(y) > 0$ and, hence, $h(y) \geq my$ for $0 < y \leq 4/3$.

Further, we define

$$F_{n,1}^{(\delta)}(q;t)=\begin{cases} -i(1-q)\displaystyle\int_0^{4/3} w_n^{(\delta)}(t;-q+i(1-q)y)\,dy & \text{for } t>0;\\ i(1-q)\displaystyle\int_0^{4/3} w_n^{(\delta)}(t;-q-i(1-q)y)\,dy & \text{for } t<0\end{cases}$$

and

$$F_{n,2}^{(\delta)}(q;t)=\begin{cases} -i(1-q)\displaystyle\int_{4/3}^{\infty} w_n^{(\delta)}(t;-q+i(1-q)y)\,dy & \text{for } t>0;\\ i(1-q)\displaystyle\int_{4/3}^{\infty} w_n^{(\delta)}(t;-q-i(1-q)y)\,dy & \text{for } t<0\end{cases}$$

as well as

$$G_{n,1}^{(\delta)}(q;t)=\begin{cases} i(n+q+1)\displaystyle\int_0^{4/3} w_n^{(\delta)}(t;n+q+i(n+q+1)y)\,dy & \text{for } t>0;\\ -i(n+q+1)\displaystyle\int_0^{4/3} w_n^{(\delta)}(t;n+q-i(n+q+1)y)\,dy & \text{for } t<0\end{cases}$$

and

$$G_{n,2}^{(\delta)}(q;t)=\begin{cases} i(n+q+1)\displaystyle\int_{4/3}^{\infty} w_n^{(\delta)}(t;n+q+i(n+q+1)y)\,dy & \text{for } t>0;\\ -i(n+q+1)\displaystyle\int_{4/3}^{\infty} w_n^{(\delta)}(t;n+q-i(n+q+1)y)\,dy & \text{for } t<0.\end{cases}$$

Then, from (2.21), (2.22) and (2.23), we obtain that

$$H_n^{(\delta)}(q;t)=\sum_{j=i}^{2} F_{n,j}^{(\delta)}(q;t)+\sum_{j=1}^{2} G_{n,j}^{(\delta)}(q;t), 0<|t|<\infty.$$

Since $\mathrm{Im}\{(1\pm iy)^{1/2}\}=\pm\varphi(y)$ for $y>0$,

$$F_{n,1}^{(\delta)}(q;t)=O\left(n^{\delta-1}\int_0^{4/3}\exp(-|t|(1-q)^{1/2}my)\,dy\right)=O(n^{\delta-1})=o(1)$$

uniformly with respect to $t\in\mathbb{R}$, and

$$F_{n,2}^{(\delta)}(q;t)=O\left(n^{\delta-1}\int_{4/3}^{\infty}\exp(-(1/2)|t|(1-q)^{1/2}y^{1/2})\,dy\right)$$

$$=O\left(n^{\delta-1}t^{-2}\int_0^{\infty}\exp(-y^{1/2})\,dy\right)=O(n^{\delta-1}t^{-2})$$

uniformly with respect to $t\in\mathbb{R}^*:=\mathbb{R}\setminus\{0\}$ when $n\to\infty$.

Since $\sup_{0<y\leq\infty} y^{1-\delta}|\Gamma(\delta-q\pm iy)/\Gamma(1-q\pm iy)|<\infty$, we have

$$|\Gamma(\delta-q\pm i(n+q+1)y)/\Gamma(1-q\pm iy)|=O(n^{\delta-1}y^{\delta-1})$$

uniformly with respect to $y\in(0,\infty)$. Therefore,

$$G_{n,1}^{(\delta)}(q;t)=O\biggl(n^{\delta}\int_0^{4/3}y^{\delta-1}\exp(-|t|(n+q+1)my)\,dy\biggr)$$
$$=O\biggl(n^{\delta/2}|t|^{-\delta}\int_0^{\infty}\exp(-my)\,dy\biggr)=O(n^{\delta/2}|t|^{-\delta})$$

and

$$G_{n,2}^{(\delta)}(q;t)=O\biggl(n^{\delta}\int_{4/3}^{\infty}\exp(-(1/2)|t|(n+q+1)^{1/2}y^{1/2})\,dy\biggr)$$
$$=O\biggl(n^{\delta-1}t^{-2}\int_0^{\infty}\exp(-y^{1/2})\,dy\biggr)=O(n^{\delta-1}t^{-2})$$

uniformly with respect to $t\in\mathbb{R}^*$ when n tends to infinity.

We define

$$W_{n,1}^{(\delta)}(q;t)=\sum_{j=1}^{4}I_{n,j}^{(\delta)}(q;t)+F_{n,1}^{(\delta)}(q;t),$$
$$W_{n,2}^{(\delta)}(q;t)=F_{n,2}^{(\delta)}(q;t)+G_{n,2}^{(\delta)}(q;t)$$

and $W_{n,3}^{(\delta)}(q;t)=G_{n,1}^{(\delta)}(q;t)$. Then $W_n^{(\delta)}(t)=\sum_{j=1}^{3}W_{n,j}^{(\delta)}(q;t)$ and, moreover, $W_{n,1}^{(\delta)}(q;t)=O(1), W_{n,2}^{(\delta)}(q;t)=O(n^{\delta-1}t^{-2})$ and $W_{n,3}^{(\delta)}(q;t)=O(n^{\delta/2}|t|^{-\delta})$ uniformly with respect to $t\in(-T,0)\cup(0,T)$ for $n\to\infty$.

(VII.2.4) *Suppose that* $0<\lambda<\infty, \alpha\in\mathbb{R}\backslash\mathbb{Z}^-$, $\delta>0$ *and let* F *be a locally* L*-integrable complex-valued function on the parabola* $p(\lambda)$. *Suppose that* F *satisfies the following condition*:

$$(2.24)\qquad \int_{p(\lambda)}|\zeta|^{\nu(\alpha)}|F(\zeta)|\,ds<\infty,\ \nu(\alpha)=1/2+\max(-1,\alpha/2-5/4).$$

If F *is continuous at the point* $z_0\in p(\lambda)$ *and the integral in* (1.28) *exists in Cauchy sence for* $z=z_0$, *then the series in the Laguerre polynomials* $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ *representing the function* [Chapter V, (3.28)] *in the region* $\Delta(\lambda)$ *is* (C,δ)*-summable at the point* z_0 *with sum* (1.28), *where* $z=z_0$.

Proof. Define

$$(2.25),\qquad \sigma_n^{(\delta)}(\alpha;z_0)=(A_n^{(\delta)})^{-1}\sum_{k=0}^{n}A_{n-k}^{(\delta-1)}S_k^{(\alpha)}(z_0),\ z_0\in p(\lambda),\ n=0,1,2\ldots,$$

i.e. $\{\sigma_n^{(\delta)}(\alpha;z_0)\}_{n=0}^{\infty}$ are the (C,δ)-means at the point z_0 for the series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ which represents the function [V, (3.28)] in the region $\Delta(\lambda)$.

Since $\sum_{k=0}^{n} A_{n-k}^{(\delta-1)} = A_n^{(\delta)}$, from (1.12) and (2.25) we obtain that

$$\sigma_n^{(\delta)}(\alpha;z_0) = \frac{1}{2\pi i}\int_{p(\lambda)} \frac{1-L_n^{(\delta)}(z_0,\zeta)}{\zeta - z_0} F(\zeta)\,d\zeta + Z_n^{(\delta,\alpha)}(z_0),$$

where $L_n^{(\delta)}(z_0,\zeta) = (A_n^{(\delta)})^{-1}\sum_{k=0}^{n} A_{n-k}^{\delta-1} E_k(z_0,\zeta)$, and

$$Z_n^{(\delta,\alpha)} = (A_n^{(\delta)})^{-1}\sum_{k=0}^{n} A_{n-k}^{(\delta-1)} r_k^{(\alpha)}(z_0),\ n=0,1,2,\dots. \tag{2.26}$$

Since $\lim_{n\to\infty} r_n^{(\alpha)}(z_0) = 0$, from the regularity of the (C,δ)-summation for $\delta>0$ and from (2.26) it follows that $\lim_{n\to\infty} Z_n^{(\delta,\alpha)}(z_0) = 0$.

Suppose that $F(z_0)=0$ and define

$$R_n^{(\delta)}(\alpha;z_0) = \frac{1}{2\pi i}\int_{p(\lambda)\setminus\gamma_n(z_0)} \frac{F(\zeta)}{\zeta - z_0}\,d\zeta,$$

where $\gamma_n(z_0) = \{\zeta\in p(\lambda) : |\zeta - z_0| < (n+1)^{-1/2}\},\ n=0,1,2,\dots.$

It is evident that if $\rho > \max(1, 2\lambda^2, |z_0|)$, then there exists a positive integer n_0 such that $\gamma_n(z_0)\subset\gamma(\lambda,\rho)$ for $n>n_0$. Then, $R_n^{(\delta)}(\alpha;z_0) = U_n^{(\delta)}(z_0) + T_{n,1}^{(\delta,\rho)}(z_0)$ $+T_{n,2}^{(\delta,\rho)}(z_0) + Z_n^{(\delta,\alpha)}(z_0)$, where

$$U_n^{(\delta)}(z_0) = \frac{1}{2\pi i}\int_{\gamma_n(z_0)} \frac{1-L_n^{(\delta)}(z_0,\zeta)}{\zeta - z_0} F(\zeta)\,d\zeta,\ n\ge n_0,$$

$$T_{n,1}^{(\delta,\rho)}(z_0) = -\frac{1}{2\pi i}\int_{\gamma(\lambda,\rho)\setminus\gamma_n(z_0)} \frac{L_n^{(\delta)}(z_0,\zeta)}{\zeta - z_0} F(\zeta)\,d\zeta,\ n\ge n_0$$

and

$$T_{n,2}^{(\delta,\rho)}(z_0) = \frac{1}{2\pi i}\int_{p(\lambda)\setminus\gamma(\lambda,\rho)} \frac{L_n^{(\delta)}(z_0,\zeta)}{\zeta - z_0} F(\zeta)\,d\zeta\, n=0,1,2,\dots. \tag{2.27}$$

Since $\operatorname{Re}\eta(z_0,\zeta) = 0$ for $\zeta\in p(\lambda)$, $|E_n(z_0,\zeta)| = 1$ for $\zeta\in p(\lambda), n=0,1,2,\dots.$ Hence, $|L_n^{(\delta)}(z_0,\zeta)|\le 1$ for $\zeta\in p(\lambda)$ and $n=0,1,2,\dots.$

If $M(\lambda,\rho) := sup\{|\zeta(\zeta-z_0)^{-1}| : \zeta\in p(\lambda)\setminus\gamma(\lambda,\rho)\}$, then (2.27) gives that

$$|T_{n,2}^{(\delta,\rho)}(z_0)| \le (2\pi)^{-1} M(\lambda,\rho)\int_{p(\lambda)\setminus\gamma(\lambda,\rho)} |\zeta|^{-1}|F(\zeta)|\,ds,\ n=0,1,2,\dots.$$

Since $\lim_{\rho\to\infty} M(\lambda,\rho) = 0$, from the last inequality it follows that if $\varepsilon > 0$ is fixed, then $\rho < \max(1, 2\lambda^2, |z_0|)$ can be chosen in such a way that $|T_{n,2}^{(\delta,\rho)}(z_0)| < \varepsilon$ for $n = 0, 1, 2, \dots$.

Now, we parametrize the parabola $p(\lambda)$ by means of the equality $\eta(z_0, \zeta) = -it$, i.e. $\zeta = \zeta(t) = z_0 - 2i(-z_0)^{1/2} + t^2, -\infty < t < \infty$. Let $\zeta_1 = \zeta(-t_1)$ and $\zeta_2 = \zeta(t_2)$ be the initial and endpoint of the arc $\gamma(\lambda, \rho)$, respectively. Denote by $\zeta_{n,1} = \zeta(-t_{n,1})$ the initial point and by $\zeta_{n,2} = \zeta(t_{n,2})$ the end point of the arc $\gamma_n(z_0)$ provided $n \geq n_0$. It is clear that $0 < t_{n,j} < t_j, j = 1, 2$. Moreover, it is easy to prove that there exists

$$\lim_{n\to\infty} (n+1)^{1/2} t_{n,j} = 0, \infty,\ \ j = 1, 2. \tag{2.28}$$

If $\omega(z_0; t) = (t - 2i(-z_0)^{1/2})^{-1} F(\zeta(t))\zeta'(t)$ for $-t_1 \leq t \leq t_2$, then

$$T_{n,1}^{(\delta,\rho)}(z_0) = -\frac{1}{2\pi i}\left(\int_{-t_1}^{-t_{n,1}} + \int_{t_{n,2}}^{t_2}\right) L_n^{(\delta,\rho)}(z_0, \zeta(t)) t^{-1} \omega(z_0; t)\, dt.$$

Since $L_n^{(\delta}) (z_0, \zeta(t)) = (A_n^{(\delta)})^{-1} W_n^{(\delta)}(-2t), n = 0, 1, 2, \dots$, we have

$$-2\pi i A_n^{(\delta)} T_{n,1}^{(\delta,\rho)}(z_0) = \sum_{j=1}^{3} \int_{-t_1}^{-t_{n,1}} W_{n,j}(q; -2t) t^{-1} \omega(z_0; t)\, dt \tag{2.29}$$

$$+ \sum_{j=i}^{3} \int_{t_{n,2}}^{t_2} W_{n,j}^{(\delta)}(q; -2t) t^{-1} \omega(z_0; t)\, dt = P_{n,1}^{(\delta)}(q; z_0) + P_{n,2}^{(\delta)}(q; z_0),$$

where

$$P_{n,1}^{(\delta)}(q; z_0) = -\sum_{j=1}^{3} \int_{t_{n,1}}^{t_1} W_{n,j}^{(\delta)}(q; 2t) t^{-1} \omega(z_0; -t)\, dt$$

and

$$P_{n,2}^{(\delta)}(q; z_0) = \sum_{j=1}^{3} \int_{t_{n,2}}^{t_2} W_{n,j}^{(\delta)}(q; -2t) t^{-1} \omega(z_0; t)\, dt.$$

provided $0 < q < \min(1/2, \delta)$.

If $\Phi(t) = \int_0^t |\omega(z_0; -\tau)|\, d\tau, 0 \leq t \leq t_1$, then from (2.18) we obtain

$$|P_{n,1}^{(\delta\rho)}(q; z_0)| \leq \sum_{j=1}^{3} \int_{t_{n,1}}^{t_1} |W_{n,j}^{(\delta)}(q; 2t) \omega(z_0; t)| t^{-1}\, dt \tag{2.30}$$

$$= O\left(\int_{t_{n,1}}^{t_1} \{t^{-1} + n^{\delta-1} t^{-3} + + n^{\delta/2} t^{-1-\delta}\} |\omega(z_0; t)|\, dt\right)$$

$$= O\left(\int_{t_{n,1}}^{t_1} \{t^{-1} + n^{\delta-1}t^{-3} + n^{\delta/2}t^{-1-\delta}\}\, d\Phi(t)\right)$$

for $n \to \infty$.

Since the function $\omega(z_0; -t)$ is continuous at the point $t = 0$ and, moreover, $\omega(z_0; 0) = 0$, it follows that $\Phi(t) = o(t)$ when t tends to zero. Hence, if $\varepsilon > 0$ is fixed, then there exists $\theta = \theta(\varepsilon) \in (0, t_1)$ such that $\Phi(t) \le \varepsilon t$ when $0 \le t \le \theta$.

If μ is real, then (2.28) yields

$$\int_{t_{n,1}}^{t_1} t^{-\mu}\, d\Phi(t) = t_1^{-\mu}\Phi(t_1) - t_{n,1}^{-\mu}\Phi(t_{n,1}) + \mu \int_{t_{n,1}}^{\theta} t^{-\mu-1}\Phi(t)\, dt + \mu \int_{\theta}^{t_1} t^{-\mu-1}\Phi(t)\, dt$$

$$= O(1) + O\left(\varepsilon \int_{t_{n,1}}^{\theta} t^{-\mu}\, dt\right) + o(t_{n,1}^{1-\mu}).$$

as $n \to \infty$.

In particular,

$$\int_{t_{n,1}}^{t_1} t^{-1}\, d\Phi(t) = O(1) + O(\varepsilon \log(n+1)),$$

$$\int_{t_{n,1}}^{t_1} t^{-3}\, d\Phi(t) = O(1) + O(\varepsilon n) + o(n)$$

and

$$\int_{t_{n,1}}^{t_1} t^{-1-\delta}\, d\Phi(t) = O(1) + O(\varepsilon n^{\delta/2}) + o(n^{\delta/2}).$$

Then from (2.30) we obtain $P_{n,1}^{(\delta,\rho)}(q; z_0) = O(1) + O(\varepsilon n^{\delta/2}) + o(n^{\delta/2})$. In the same way, but using the function $\Psi(t) = \int_0^t |\omega(z_0; \tau)|\, d\tau, 0 \le t \le t_2$, we find that $P_{n,2}^{(\delta,\rho)}(q; z_0) = O(1) + O(\varepsilon n^{\delta}) + o(n^{\delta})$. Since $A_n^{(\delta)} = (\Gamma(\delta+1))^{-1} n^{\delta}(1+o(1)), n \to \infty$, from (2.29) we obtain $T_{n,1}^{(\delta,\rho)}(z_0) = O(\varepsilon)$ when n is large enough.

Further, the inequality

$$|t^{-1}(1 - L_n^{(\delta)}(z_0; \zeta(t))| \le (A_n^{(\delta)})^{-1} \sum_{k=0}^{n} A_{n-k}^{(\delta-1)} |t^{-1}(1 - \exp(-2it\sqrt{k+1}))|$$

holds for $t \ne 0$ and, hence,

$$|U_n^{(\delta)}(z_0)| \le \pi^{-1}(n+1)^{1/2} \int_{-t_{n,1}}^{t_{n,2}} |\omega(z_0, t)|\, dt = O((n+1)^{1/2}(\Phi(t_{n,1}) + \Psi(t_{n,2})))$$

$$= o((n+1)^{1/2}(t_{n,1} + t_{n,2})) = o(1),$$

when n tends to infinity.

So far the validity of the assertion we are to prove is established when $F(z_0)$ $= 0$. In the general case, we use the function $H(\zeta) = F(\zeta) - F(z_0)\exp(z_0 - \zeta)$. It is clear that it satisfies the same conditions as the function F and, moreover, $H(z_0) = 0$. Therefore, if $\delta > 0$, then the series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}, \alpha \in \mathbb{R} \setminus \mathbb{Z}^-$, which represents the function

$$h(\zeta) = \frac{1}{2\pi i}\int_{p(\lambda)} \frac{H(\zeta)}{\zeta - z}\,d\zeta,\ \ z \in \mathbb{C} \setminus \imath(\lambda)$$

in the region $\Delta(\lambda)$, is (C, δ)-summable at the point z_0 with (C, δ)-sum

$$h(z_0) = \frac{1}{2\pi i}\int_{p(\lambda)} \frac{H(\zeta)}{\zeta - z_0}\,d\zeta = \frac{1}{2\pi i}\int_{p(\lambda)} \frac{F(\zeta)}{\zeta - z_0}\,d\zeta - \frac{1}{2}F(z_0).$$

Remark. It is easy to prove that for $z_0 \in p(\lambda)$ we have

$$\frac{1}{2\pi i}\int_{p(\lambda)} \frac{\exp(z_0 - \zeta)}{\zeta - z_0}\,d\zeta = \frac{1}{2}$$

in the Cauchy sence.

Since the entire function $\exp(z_0 - z)$ is in the space $\mathcal{L}(\infty)$ for $z_0 \in \mathbb{C}$, as a corollary of **(V.3.7)** we can assert that it is representable in the whole complex plane as a convergent series in the Laguerre polynomials with a real parameter $\alpha \neq -1, -2, -3, \ldots$.

We have $f(z) = h(z) + F(z_0)\exp(z_0 - z)$ for $z \in \Delta(\lambda)$. Since the (C, δ)-summation is regular for $\delta > 0$, the series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ representing the function f in the region $\Delta(\lambda)$ is (C, δ)-summable at the point z_0 with (C, δ)-sum

$$f(z_0) = h(z_0) + F(z_0) = \frac{1}{2\pi i}\int_{p(\lambda)} \frac{F(\zeta)}{\zeta - z_0}\,d\zeta + \frac{1}{2}F(z_0).$$

As a corollary of **(VII.2.4)** we obtain the following proposition of Fejer type:

(VII.2.5) *Suppose that $0 < \lambda < \infty, \alpha \in \mathbb{R} \setminus \mathbb{Z}^-$, and letf be a complex-valued function which is continuous on $\overline{\Delta(\lambda)}$ and holomorphic in $\Delta(\lambda)$. Suppose that $|f(z)| = O(|z|^{\mu})$ for some $\mu < 1/2$ when $z \to \infty$ in $\overline{\Delta(\lambda)}$ and*

$$\int_{p(\lambda)} |\zeta|^{\nu(\alpha)}|f(\zeta)|\,ds < \infty,\ \ \nu(\alpha) = 1/2 + \max(-1, \alpha/2 - 5/4).$$

If $\delta > 0$, then the series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ representing the function f in the region $\Delta(\lambda)$ is (C, δ)-summable at every point $z_0 \in p(\lambda)$ with (C, δ)-sum $f(z_0)$.

Proof. Under the assumptions about the growth of the function f as $z \to \infty$ in $\overline{\Delta(\lambda)}$ it is easy to prove that

$$f(z) = \frac{1}{2\pi i} \int_{p(\lambda)} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

for $z \in \Delta(\lambda)$. Moreover, for $z_0 \in p(\lambda)$

$$\frac{1}{2\pi i} \int_{p(\lambda)} \frac{f(\zeta)}{\zeta - z_0} = \frac{1}{2} f(z_0).$$

in Cauchy sence.

3. Fatou type theorems

3.1 Let

$$\sum_{n=0}^{\infty} c_n z^n \tag{3.1}$$

be a power series with non zero and finite radius of convergence r. Denote by φ its sum in the disk $U(0; r)$. A point $z_0 \in C(0; r)$ is called regular for the function φ if there exist a neighbourhood $U(z_0; \rho)$ and a function $\tilde{\varphi}_{z_0} \in \mathcal{H}(U(z_0; \rho))$ such that $\tilde{\varphi}_{z_0}(z) = \varphi(z)$ for $z \in U(0; r) \cap U(z_0; \rho)$. From this definition it follows that the set of the regular points of the series (3.1) such that $0 < r = (\limsup_{n\to\infty} |c_n|^{1/n})^{-1} < \infty$ is an open subset of the circle $C(0; r)$ with respect to the relative topology on $C(0; r)$, i.e. the topology induced by that of $\mathbb{C}$.

The power series $\displaystyle\sum_{n=0}^{\infty} z^n$ is divergent at each point of the circle $C(0; 1)$ regardless of the fact that all the points of this circle, except $z = 1$, are regular for its sum. The series $\displaystyle\sum_{n=1}^{\infty} n^{-2} z^n$ is (absolutely) convergent at each point of the circle $C(0; 1)$ but nevertheless one of them, namely $z = 1$, is a singular (i.e. not regular) point for its sum. In other words, in general, there is no relation between the convergence (divergence) of a power series at points on the boundary of its disk of convergence and the regularity (singularity) of its sum at such points. But under additional conditions on the sequence $\{c_n\}_{n=0}^{\infty}$ such a relation do exists. More precisely, the following proposition holds:

(VII.3.1) *If* $0 < r = (\limsup_{n\to\infty} |c_n|^{1/n})^{-1} < \infty$ *and* $\lim_{n\to\infty} c_n r^n = 0$, *then the series* (3.1) *is uniformly convergent on each closed arc* $\gamma \subset C(0; r)$ *provided that all points of* γ *are regular for its sum.*

Let us point out that under the hypothesis of the above assertion there exist a region $D \supset \gamma$ and a function $\tilde{\varphi} \in \mathcal{H}(D)$ such that $\tilde{\varphi}(z) = \varphi(z)$ for $z \in D \cap U(0; r)$.

It means that the function $\tilde{\varphi}$ is an analytical continuation of the function φ outside the disk $U(0;r)$. Moreover, as it not difficult to see, the series (3.1) converges on that (open) arc $\tilde{\gamma} \subset C(0;r)$ which contains γ and is included in the region D. Then Abel's theorem, i.e. the proposition **(IV.4.11)** yields that the sum of the series (3.1) is $\tilde{\varphi}(z)$ for each $z \in \tilde{\gamma}$. Therefore, we may assume that the power series (3.1) represents the function φ even on the arc $\tilde{\gamma}$.

3.2 Propositions like **(VII.3.1)** hold also for series in the Laguerre and Hermite systems.

(VII.3.2) *Suppose that $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$ and let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers such that*

$$0 < -\limsup_{n\to\infty} (2\sqrt{n})^{-1} \log|a_n| = \lambda_0 < \infty \tag{3.2}$$

and

$$\lim_{n\to\infty} n^{\alpha/2+1/4} \exp(2\lambda_0) a_n = 0. \tag{3.3}$$

If every point of a finite and closed arc $\gamma \subset p(\lambda_0)$ is regular for the sum of the series [IV, (2.1)], *then this series is uniformly convergent on γ.*

Proof. Denote by f the sum of the series [IV, (2.1)] and define the function P in the region $T(\lambda_0) = \{\zeta \in \mathbb{C} : |\operatorname{Re}\zeta| < \lambda_0\}$ by the equality $P(\zeta) = f(-\zeta^2)$. There exists a segment $[\lambda_0 + ib_1, \lambda_0 + ib_2], b_1 < b_2$, every point of which is regular for the function P. Suppose that $\delta > 0$ and denote $\zeta_1 = \lambda_0 + (b_1 - \delta)i, \zeta_2 = \lambda_0 + (b_2 + \delta)i$. Let $R(\delta; \zeta_1, \zeta_2)$ be the rectangle with vertices at the points $\zeta_1 \pm \delta, \zeta_2 \pm \delta$. If δ is small enough, then there exists a simply connected region G containing the strip $T(\lambda_0)$ as well as the rectangle $R(\delta; \zeta_1, \zeta_2)$ together with its interior and such that the function P is holomorphic in G. More precisely, it means that P has a single valued analytical continuation in G.

Define

$$P_k(\zeta) = P(\zeta) - \sum_{n=0}^{k} a_n L_n^{(\alpha)}(-\zeta^2),$$

$$\omega_k(\zeta) = (\zeta - \zeta_1)(\zeta - \zeta_2) P_k(\zeta) \exp\{-2(\zeta - \lambda_0)\sqrt{k+1}\}$$

for $z \in G,\ k = 0, 1, 2, \dots,$ and

$$\alpha_k = k^{\alpha/2+1/4} \exp(2\lambda_0\sqrt{k}) a_k,\ k = 1, 2, 3 \dots. \tag{3.4}$$

In order to prove that the sequence $\left\{\sum_{n=0}^{k} a_n L_n^{(\alpha)}(-\zeta^2)\right\}_{k=0}^{\infty}$ is uniformly convergent on the segment $[\lambda_0 + ib_1, \lambda_0 + ib_2]$, it is sufficiently to show that the sequence $\{\omega_k(\zeta)\}_{k=1}^{\infty}$ tends uniformly to zero on the rectangle $R(\delta; \zeta_1, \zeta_2)$. Observe that if

$\varepsilon > 0$, then, due to the assumption (3.4), there exists a positive integer $\nu_0 = \nu_0(\varepsilon)$ such that $|\alpha_k| < \varepsilon$ when $k > \nu_0$. We have to consider the following cases:

(a) $\zeta \in [\zeta_1 - \delta, \zeta_2 - \delta]$. In this case $P(\zeta) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(-\zeta^2)$ and the asymptotic formula [III (2.3)] as well as (3.4) yield that for $k > \nu_0$

$$\omega_k(\zeta) = O\Bigg(\exp(2\delta\sqrt{k+1}) \sum_{n=k+1}^{\infty} |\alpha_n| n^{-1/2} \exp(-2\delta\sqrt{n})\Bigg)$$

$$= O(\varepsilon(2\delta\sqrt{k+1})) \sum_{n=k+1}^{\infty} n^{-1/2} \exp(-2\delta\sqrt{n})$$

$$= O(\varepsilon \exp(2\delta\sqrt{k+1})) \int_k^{\infty} t^{-1/2} \exp(-2\delta\sqrt{t})\, dt$$

$$= O(\varepsilon \exp(2\delta\sqrt{k+1})) \exp(-2\delta\sqrt{k}) = O(\varepsilon).$$

(b) $\zeta \in [\zeta_1 - \delta, \zeta_1)$ i.e. $\zeta = \xi + i\eta_1$, where $\eta_1 = \operatorname{Im}\zeta_1$ and $\lambda_0 - \delta \le \xi < \lambda_0$. In this case we have

$$\omega_k(\zeta) = \Bigg(\varepsilon \exp\{2(\lambda_0 - \xi)\sqrt{k+1}\}(\lambda_0 - \xi) \sum_{n=k+1}^{\infty} n^{-1/2} \exp\{-2(\lambda_0 - \xi)\sqrt{n}\}\Bigg)$$

$$= o\Bigg(\varepsilon \exp\{2(\lambda_0 - \xi)\sqrt{k+1}\}(\lambda_0 - \xi) \int_k^{\infty} t^{-1/2} \exp\{-2(\lambda_0 - \xi)\sqrt{t}\}\, dt\Bigg)$$

$$= O(\varepsilon \exp\{2\delta(\sqrt{k+1} - \sqrt{k})\}) = O(\varepsilon),\ \ k > \nu_0.$$

(c) $\zeta \in (\zeta_1, \zeta_1 + \delta]$ i.e. $\zeta = \xi + i\eta_1$ and $\lambda_0 < \xi \le \lambda_0 + \delta$. Then,

$$|\omega_k(\zeta)| \le \exp\{-2(\xi - \lambda_0)\sqrt{k+1}\}(\xi - \lambda_0)|\zeta - \zeta_2| \Bigg\{ |P(\zeta) - a_0|$$

$$+ \sum_{n=1}^{\nu_0} |a_n L_n^{(\alpha)}(-\zeta^2)| + \sum_{n=\nu_0+1}^{k} |a_n L_n^{(\alpha)}(-\zeta^2)| \Bigg\}$$

$$= O(\exp\{-2(\xi - \lambda_0)\sqrt{k+1}\}(\xi - \lambda_0))$$

$$+ O\Bigg(\varepsilon \exp\{-2(\xi - \lambda_0)\sqrt{k+1}\}(\xi - \lambda_0) \sum_{n=\nu_0+1}^{k} n^{-1/2} \exp\{2(\xi - \lambda_0)\sqrt{n}\}\Bigg).$$

Before going on, we need the following auxiliary proposition:

(VII.3.3) *Suppose that $a(t), 0 < t < \infty$, is a real and positive function satisfying one of the following conditions*: (i) $a(t)$ *decreases in* $(0, \infty)$; (ii) $a(t)$

increases in $(0, \infty)$; (iii) *there exists* $\tau \in (0, \infty)$ *such that* $a(t)$ *decreases in* $(0, \tau]$ *and increases in* $[\tau, \infty)$. *Then*

$$\sum_{n=1}^{k} a(n) \leq a(1) + a(k) + \int_1^k a(t)\, dt, \ \ k = 1, 2, 3, \ldots .$$

The proof is rather elementary and we leave it is as an exercise to the reader.

From **(VII.3.3)** we obtain easily that

$$\sum_{n=\nu_0+1}^{k} n^{-1/2} \exp\{2(\xi - \lambda_0)\sqrt{n}\} \leq \mathrm{Const}(\delta)(\xi - \lambda_0)^{-1} \exp\{2(\xi - \lambda_0)\sqrt{k}\}$$

provided $0 < \xi - \lambda_0 \leq \delta$ and $k \geq \nu_0 + 1$. Hence, in our case

$$\omega_k(\zeta) = O(\exp\{-2(\xi - \lambda_0)\sqrt{k+1}\}(\xi - \lambda_0)).$$

It is easy to prove that the sequence $\{\exp\{-2(\xi - \lambda_0)\}\sqrt{k+1}\}_{k=1}^{\infty}$ tends uniformly to zero when $\lambda \leq \xi \leq \lambda_0 + \delta$. Indeed, if $0 < \varepsilon < \delta$ and $\lambda_0 \leq \xi \leq \lambda + \varepsilon$, then $\exp\{-2(\xi - \lambda_0)\sqrt{k+1}\}(\xi - \lambda_0) \leq \varepsilon, \ k = 1, 2, 3 \ldots .$

If $\lambda + \varepsilon \leq \xi \leq \lambda_0 + \delta$, then $\exp\{-2(\xi - \lambda_0)\sqrt{k+1}\}(\xi - \lambda_0) \leq \delta \exp(-2\varepsilon\sqrt{k+1})$ and, hence, $\exp\{-2(\xi - \lambda_0)\sqrt{k+1}\}(\xi - \lambda_0) \leq \delta \exp(-\delta/\varepsilon) < \delta(1 + \delta/\varepsilon)^{-1} < \varepsilon$.

(d) $\zeta \in [\zeta_1 + \delta, \zeta_2 + \delta]$ i.e. $\zeta = \lambda_0 + \delta + i\eta$ and $b_1 - \delta \leq \eta \leq b_2 + \delta$. In this case we obtain

$$|\omega_k(\zeta)| \leq \exp(-2\delta\sqrt{k+1})|(\zeta - \zeta_1)(\zeta - \zeta_2)| \Bigg\{ |P(\zeta) - a_0|$$

$$+ \sum_{n=1}^{\nu_0} |a_n L_n^{(\alpha)}(-\zeta^2)| + \sum_{n=\nu_0+1}^{k} |a_n L_n^{(\alpha)}(-\zeta^2)| \Bigg\}$$

$$= O\Bigg\{ \exp(-2\delta\sqrt{k+1}) + \varepsilon \exp(-2\delta\sqrt{k+1}) \sum_{n=\nu_0+1}^{k} n^{-1/2} \exp(2\delta\sqrt{n}) \Bigg\}$$

$$= O(\exp(-2\delta\sqrt{k+1})) + O(\varepsilon)$$

uniformly with respect to ζ.

In the same way we conclude that the sequence $\{\omega_k(\zeta)\}_{k=1}^{\infty}$ tends uniformly to zero on the segments $[\zeta_2 - \delta, \zeta_2)$ and $(\zeta_2, \zeta_2 + \delta]$. Since $\omega_k(\zeta_1) = \omega_k(\zeta_2) = 0$ for $k = 1, 2, 3 \ldots$, it follows that $\lim_{k \to \infty} \omega_k(\zeta) = 0$ uniformly on the rectangle $R(\delta; \zeta_1, \zeta_2)$, Thus, the assertion is proved.

Examples:

(a) Suppose that the function F satisfies the assumption of **(V.3.10)** and, moreover, that $F(t) = O(|t|^{\delta})$ for some $\delta > 0$ when t tends to infinity. Suppose, in addition, that F has a locally L-integrable derivative in the interval $(-\infty, -\lambda_0]$ and

$$\int_{-\infty}^{-\lambda_0} |t|^{\alpha/2+1/4}|F'(t)|\,dt < \infty.$$

Then using the relation $(M_{n-1}^{(\alpha+1)}(z))' = nM_n^{\alpha}(z),\ n = 1, 2, 3, \dots,$ [Chapter I, Exercise. 25] and [Chapter V, (3.20)], we find that

$$a_n = (2\pi i I_n^{\alpha})^{-1}\Big\{M_{n-1}^{\alpha+1}(-\lambda_0^2)F(-\lambda_0^2) - \int_{-\infty}^{-\lambda_0} M_{n-1}^{\alpha+1}(t)F'(t)\,dt\Big\},\ n = 1, 2, 3 \dots.$$

Further, the asymptotic formula [Chapter III, (3.1)], the inequality [Chapter III,(5.1)], and Stirling's formula yield that $n^{\alpha/2+1/4}\exp(2\lambda_0\sqrt{n})a_n = O(n^{-1/2})$, $n \to \infty$. Hence, the series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ which represents the function [V, (3.19)] in the region $\Delta(\lambda_0)$ is uniformly convergent on each finite and closed arc of the parabola $p(\lambda_0)$ not containing the point $-\lambda_0^2$.

(b) Suppose that $0 < \lambda_0 < \infty, \alpha > -1$, and let k be a positive integer. Since the function $(z + \lambda_0^2)^{-k-1}$ is in the space $\mathcal{L}(\lambda_0)$, from **(V.3.7)** it follows that for $z \in \Delta(\lambda_0)$ it has a representation by a series in the Laguerre polynomials with parameter α. Moreover, **(IV.5.3)** and [I, (3.1)] yield

$$\begin{aligned} a_n &= \frac{1}{I_n^{\alpha}}\int_0^{\infty} t^{\alpha}\exp(-t)L_n^{(\alpha)}(t)(t+\lambda_0^2)^{-k-1}\,dt \\ &= \frac{\Gamma(n+k+1)}{\Gamma(k+1)\Gamma(n+\alpha+1)}\int_0^{\infty}\frac{t^{n+\alpha}\exp(-t)}{(t+\lambda_0^2)^{n+k+1}}\,dt \\ &= -\frac{\Gamma(n+k+1)}{\Gamma(k+1)\Gamma(n+k+1)}M_{n+k}^{(\alpha-k)}(-\lambda_0^2),\ n = 0, 1, 2 \dots. \end{aligned}$$

Further, from the asymptotic formulas [Chapter III, (2.3)], [Chapter III, (3.1)] as well as from Stirling's formula one can obtain that $|a_n L_n^{(\alpha)}(z)| = A(k, \alpha, \lambda_0)n^{k/2-1/4}\{1 + l_n^{(\alpha)}(z)\}$, $n = 1, 2, 3 \dots,$ where $A(k, \alpha, \lambda_0) \neq 0$ and $l_n^{(\alpha)}(z) = o(1)$ when n tends to infinity provided that $z \in p(\lambda_0)$ and $z \neq -\lambda_0^2$. Therefore, the series in Laguerre polynomials with parameter α, representing the function $(z+\lambda_0)^{-k-1}$ in the region $\Delta(\lambda_0)$, diverges at each point $z \in p(\lambda_0)\setminus\{-\lambda_0^2\}$ although each such point is regular for its sum. Observe that our second example does not contradict to **(VII.3.2)** since in this case the condition (3.3) is not satisfied.

(VII.3.4) *Suppose that* $0 < \mu_0 < \infty$ *and* $\alpha \in \mathbb{R}\setminus\mathbb{Z}^-$. *If*

$$\lim_{n\to\infty} n^{\alpha/2+1/4}\exp(-2\mu_0\sqrt{n})b_n = 0,$$

then the series [IV, (2.5)] *is uniformly convergent on every finite and closed arc of the parabola* $p(\mu_0)$ *consisting of points which are regular for its sum.*

The above proposition can be proved by the same techniques which were used in the proof of **(VII.3.2)**. That is why we shall only illustrate it by a suitable example.

Suppose that F is a differentiable function on the ray $[-\mu_0^2, \infty)$ and let F satisfy the conditions of the proposition **(V.4.3)**. Suppose that F' is locally L-integrable and, moreover, that $F'(t) = O(t^\gamma \exp(-t/2))$ for some $\gamma < \alpha/2 - 1$ when t tends to infinity. Then by means of [IV, (5.8)] we obtain the following representations for the coefficients [V, (4.10)]

$$b_n = (2\pi i I_n^{(\alpha)})^{-1}\left\{L_{n+1}^{(\alpha-1)}(-\mu_0^2)F(-\mu_0^2) + \int_{-\mu_0^2}^{\infty} L_{n+1}^{(\alpha-1)}(t)F'(t)\,dt\right\},\ n = 0, 1, 2 \ldots.$$

Further, from Stirling's formula, the asymptotic formulas [Chapter III, (2.4)], [Chapter III, (4.1)] as well as from the estimate ($\omega \geq 1$)

$$\max_{t\in[-\mu_0^2,\omega]} |L_n^{(\alpha)}(t)| = O(n^{\alpha/2-1/4}\exp(2\mu_0\sqrt{n})),\ n \to \infty,$$

which was used in the proof of **(V.4.3)**, we obtain

$$n^{\alpha/2+1/4}\exp(-2\mu_0\sqrt{n})b_n = O(n^{-1/2}),\ n \to \infty.$$

Hence, the series expansion in the region $\Delta^*(\mu_0)$ of the function [V, (4.9)] in the Laguerre associated functions with parameter α is uniformly convergent on each finite and closed arc of the parabola $p(\mu_0)$ not containing the point $-\mu_0^2$.

Fatou type propositions hold also for series in the Hermite systems. They can be proved in the same way as those for the Laguerre systems. That is why we only state them below without proofs:

(VII.3.4) *If* $0 < \tau_0 < \infty$ *and* $\lim_{n\to\infty} n^{1/2}(2n/e)^{n/2}\exp(\tau_0\sqrt{2n+1})a_n = 0$, *then the series* [IV, (3.1)] *is uniformly convergent on each segment* $\sigma \subset \partial S(\tau_0)$ *consisting of points which are regular for its sum.*

(VII.3.5) *If* $0 < \tau_0 < \infty$ *and* $\lim_{n\to\infty} n^{1/2}(2n/e)^{n/2}\exp(-\tau_0\sqrt{2n+1})b_n = 0$, *then each of the the series* [IV, (3.2)] *is uniformly convergent on every segment* $\sigma \subset \partial S^*(\tau_0)$ *consisting of regular points for its sum.*

Remark. Notice that, in general, the last two propositions are not corollaries of **(VII.3.2)** and **(VII.3.3)** *via* the relations between the Laguerre and Hermite systems. Indeed, if $z_0 \in \partial S(\tau_0)$ is a regular point for the function $f(z) = \sum_{n=0}^{\infty} a_n H_n(z)$, then it does not need to be a regular point for the functions $g(z) = \sum_{n=0}^{\infty} a_{2n}H_{2n}(z)$ and $h(z) = \sum_{n=0}^{\infty} a_{2n+1}H_{2n+1}(z)$. Here is an example

$(\tau_0 = 1, z_0 = i)$:

$$f(z) = \frac{1}{1 - iz} = \frac{1}{1 + z^2} + \frac{iz}{1 + z^2}.$$

Exercises

1. State an prove a proposition like **(VII.1.4)** for series representations by means of Jacobi associated functions.

2. Suppose that $0 < \lambda_0 < \infty, \alpha > -1$ and let $f(z)$ be a complex-valued function which is continuous for $\operatorname{Re} z >\geq -\lambda_0^2$ and holomorphic for $\operatorname{Re} z > -\lambda_0^2$ and such that:

$$(a) \lim_{\operatorname{Re} z \geq -\lambda_0^2, z \to \infty} = 0;$$

$$(b) \int_{-\infty}^{\infty} |f(-\lambda_0^2 + i\tau)|\, d\tau < \infty.$$

Prove that f for $z \in \Delta(\lambda_0)$ is representable by a series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ and, moreover, that this series converges to $f(z_0)$ at every point $z_0 \in p(\lambda_0) \setminus \{-\lambda_0\}$.

If in addition

$$(c) \int_{p(\lambda_0)} |f(\zeta)\zeta^{-1}|\, ds < \infty$$

and

$$(d) \int_{-\rho}^{\rho} |\tau^{-1}\{f(-\lambda_0 + i\tau) - f(-\lambda_0)\}|\, d\tau < \infty,\ \rho > 0,$$

then the series, just mentioned, converges at the point $-\lambda_0^2$ with sum $f(-\lambda_0^2)$.

3. State and prove a proposition like **(VII.1.8)** for series representations by means of Laguerre associated functions.

4. State and prove a proposition like **(VII.1.8)** for series representations by means of Hermite polynomials.

5. State and prove a proposition like **(V.1.8)** for series representations by means of Hermite associated functions.

6. State and prove a proposition like **(VII.2.2)** for series representations by means of Jacobi associated functions.

7. Suppose that $0 < \lambda_0 < \infty, \alpha > -1$ and let the complex-valued function f satisfy the conditions (a), (b) and (c) of Exercise 2. Then the series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ which represents f in the region $\Delta(\lambda_0)$ is $(C,1)$-summable at the point $-\lambda_0^2$ with $(C,1)$-sum $f(-\lambda_0^2)$.

8. Replace the $(C,1)$-summation in Exercise 7 by (C,δ)-summation provided $\delta > 0$.

9. State and prove a proposition like **(VII.1.8)** for series representations by means of Hermite polynomials.

10. State and prove a proposition like **(VII.1.8)** for series representations by means of Hermite associated functions.

11. State and prove theorems of Fatou type for series in:

(a) Laguerre associated functions;

(b) Hermite polynomials;

(c) Hermite associated functions.

Comments and references

The proposition **(VII.1.2)** is due to G. Boychev [1,2]. Its "local" version, i.e when the closed arc $\tilde{e}(r)$ reduces to a point, is proved by P. Rusev [7].

At the end of the paper [I. Baičev, 2] it is pointed out that Cesaro summability of series in Jacobi polynomials can be established by the same techniques which is applied in the paper just mentioned to series in generalized Bessel polynomials.

For convergence and Cesaro summability of Laguerre series on the boundaries of their regions of convergence we refer to [P. Rusev, 27,28] and [G. Boychev and P. Rusev, 1].

The proof of the (C,δ)-statement **(VII.1.2)** is given in [G. Boychev, 2] and **(VII.1.3)** is announced in [G. Boychev, 3].

Let the complex-valued function φ satisfy a Hölder condition with an exponent $\gamma \in (0,1]$ on the whole ellipse $e(r), 1 < r < \infty$. Then, as it is proved by G. Boychev [9], for the partial sums $\{S_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ of the series in the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ which represents the function [Chapter V, (1.1)] in the region $E(r)$ we have that

$$(*) \qquad S_n^{(\alpha,\beta)}(z) - \Phi(z) = O(n^{-\gamma}\log n)$$

uniformly with respect to $z \in e(r)$ when n tends to infinity.

Moreover, if $\gamma > 1/2$, then the series just mentioned is absolutely convergent on $e(r)$ and uniformly (C,δ)-summable for each $\delta > -\gamma$ [G. Boychev, 6].

Suppose that the complex-valued function f is continuous on the closed region $\overline{E(r)}$, and holomorphic in $E(r)$, and that it satisfies a Hölder condition with an exponent $\gamma \in (0,1]$ on the ellipse $e(r)$. Denote by $\{S_n^{(\alpha,\beta)}(f;z)\}_{n=0}^{\infty}$ the partial sums of the series in Jacobi polynomials with parameters α and β representing the function f in $E(r)$. Then as a corollary of $(*)$ we have

$$(**) \qquad S_n^{(\alpha,\beta)}(f;z) - f(z) = O(n^{-\gamma}\log n),\ n \to \infty$$

uniformly on $e(r)$ and, hence, on $\overline{E(r)}$.

Moreover, the series just mentioned is absolutely convergent on $e(r)$ provided that $\gamma > 1/2$ and uniformly (C,δ)-summable on $e(r)$ for $\delta > -\gamma$.

Notice that $(**)$ generalizes a particular case of a proposition about the rate of convergence of expansions of holomorphic functions in series of Chebyshev polynomials [P.K. Suetin, 1, p.95, (11)].

In [L. Boyadjiev, 10] it is proved that if $1/2 < \delta < 1$ and the series [Chapter IV, (3.1)] is (C, δ)-summable at a point w_0 with $\operatorname{Im} w_0 > 0$, then it is uniformly (C, δ)-summable on each set of the kind $S(w_0, \eta, \varphi)$. This assertion can be considered as a (C, δ)-version of the corresponding theorem of Abel's type **(IV.4.5)**.

Let $\{\lambda_n\}_{n=0}^{\infty}$ be the sequence defined in **(III.2.1)** and let

$$c_n(z) = \lambda_n \cos\{(2n+1)^{1/2} z - n\pi/2\},\ n = 0, 1, 2, \ldots.$$

As it is proved by E. Hille [2, Theorem 2.4], the series [Chapter IV, (3.1)] converges at a point $z_0 \in \mathbb{C} \setminus \mathbb{R}$ if an only if the series $\sum_{n=0}^{\infty} c_n(z)$ converges at this point. This is nothing but an equiconvergence of a series in Hermite polynomials and the corresponding Fourier series.

Equiconvergence or even (C, δ)-equisummability in the sence of E. Hille is established by L. Boyadjiev [1,9,10] for series in Laguerre's polynomials. Similar results are given by G. Boychev [10].

For the series (2.1) it is said that it is B-summable with sum s if there exists

$$\lim_{t \to \infty} \exp(-t) \sum_{n=0}^{\infty} \frac{s_n t^n}{n!} = s,$$

where $s_n = \sum_{\nu=0}^{n} u_\nu,\ n = 0, 1, 2, \ldots.$ If

$$\int_0^{\infty} \exp(-t) \left(\sum_{n=0}^{\infty} \frac{u_n t^n}{n!} \right) dt = s,$$

then the series (2.1) is called B'-summable

Remark. The above methods of summation are known as Borel's methods [G.H. Hardy, 1].

Suppose that z_0 is a point on the ellipse $e(r), 1 < r < \infty$ and that the estimate $\Phi(t) = O(t(-\log|t|)^{-1})$ holds for the function (2.7) when t tends to zero. Then the series in Jacobi polynomials which represents the function [Chapter V, (1.1)] in the region $E(r)$ is B-summable at the point z_0 with sum [Chapter V, (1.10)], where $z = z_0$ [G. Boychev, 7].

It is well-known that if the power series (3.1) has finite and non-zero radius of convergence r, then it is B'-summable at each point $z_0 \in C(0; r)$ which is regular for its sum. A proposition like the last one is proved by G.Boychev [8] for series in the ultraspherical polynomials.

It seems that the classical Fatou theorem **(VII.3.1)** holds for series in more general (denumerable) systems of holomorphic functions. E.g. in [I. Baičev, 3] its validity is proved for series in the generalized Bessel polynomials.

A theorem of Fatou type holds also for series in Jacobi polynomials. Moreover, as it is proved in [G.Boychev, 11], if $1 < (\limsup_{n\to\infty} |a_n|^{1/n})^{-1} = r < \infty$ and $a_n r^n = o(n^\delta)$ for some $\delta > -1/2$, then the series [Chapter IV, (1.2)] is $(C, \delta - 1/2)$-summable on every closed arc $\tilde{e}(r) \subset e(r)$ consisting of regular points for its sum. The last statement is nothing but a theorem of Fatou-Riesz type for series in Jacobi polynomials.

As for series in Laguerre polynomials $\{L_n^{(\alpha)}\}_{n=0}^{\infty}$ is concerned, it is rather surprising that the corresponding Fatou condition (3.3) involves the parameter α. But the proof of **(VII.3.2)**, as it is given in [P. Rusev, 12], follows the same ideas as in the classical case of power series.

Addendum

A REVIEW ON SINGULAR POINTS AND ANALYTICAL CONTINUATION OF SERIES IN THE CLASSICAL ORTHOGONAL POLYNOMIALS

1. Singular points and analytical continuation of series in Jacobi polynomials

1.1 It seems that one of the earliest paper on singular points of an analytic function defined by a series in a system of classical orthogonal polynomials is that of G. Faber [1]. In this paper is studied the relation between the singular points of the analytic function defined by a series of the kind

$$\sum_{n=0}^{\infty} a_n P_n(z), \tag{1.1}$$

where $\{P_n(z)\}_{n=0}^{\infty}$ are the Legendre's polynomials, and the singular points of the analytic function defined by the power series

$$\sum_{n=0}^{\infty} a_n w^n. \tag{1.2}$$

The same problem is considered later by Z. Nehari [1].

Suppose that $\mathcal{P}$ is an analytic function in the sence of Weirestrass and let P_a be an element of $\mathcal{P}$, i.e. P_a is a convergent power series with center at the point $a \in \mathbb{C}$. Suppose that there exists a path $\Gamma : z = \gamma(t), 0 \leq t \leq 1$ starting at the point a and such that for each $\delta \in (0,1)$ the element P_a is continuable along the path $\Gamma_\delta : z = \gamma(t), 0 \leq t \leq \delta$ but not along Γ. In such a case we say that $b = \gamma(1)$ is a singular point of the function $\mathcal{P}$ which is generated by the element P_a along the path Γ.

If

$$1 < R = \{\limsup_{n\to\infty} |a_n|^{1/n}\}^{-1} < \infty, \tag{1.3}$$

then by **(II.1.1),(b)** the series (1.1) is convergent in the region $E(R)$ and, by the classical Cauchy-Hadamard formula, R is the radius of convergence of the series (1.2).

Denote by $\mathcal{F}$ the analytic function defined by the sum of the series (1.1), and let $\mathcal{A}$ be the analytic function defined by the power series (1.2). By P and A we denote the elements of $\mathcal{F}$ and $\mathcal{A}$ centered at the point $z = w = 1$, respectively.

Denote by J the Zhukovskii transformation, i.e.

$$J(w) = \frac{1}{2}(w + w^{-1}). \tag{1.4}$$

If Γ is a path in the z-plane, then there exist two paths Γ' and Γ'' in the w-plane such that $J(\Gamma') = J(\Gamma'') = \Gamma$, i.e. Γ has two pre-images by the transformation (1.4). NEHARI'S theorem is the following proposition:

(Add.1.1) *A point $\tau \neq \pm 1$ is singular for the function $\mathcal{F}$ generated by the element P along a path which starts at $z = 1$ and avoids the points 1 and -1 if and only if*

$$\tau = \frac{1}{2}(\sigma + \sigma^{-1}), \tag{1.5}$$

where σ is a singular point of the function $\mathcal{A}$, which is generated by the element A along the path which starts at $w = 1$ and coincides with one of the pre-images of Γ by the transformation (1.4)

As it is proved by NEHARI [1], if f is the sum of the series (1.1) and g is that of (1.2), then

$$g(w) = \frac{1}{2\pi i} \int_{C_\varepsilon} \frac{f(z)\, dz}{(1 - 2wz + z^2)^{1/2}} \tag{1.6}$$

and

$$f(z) = \frac{z^2 - 1}{2\pi i} \int_L \frac{g(w)\, dw}{(1 - 2zw + w^2)^{1/2}}, \tag{1.7}$$

where C_ε is the circle $|z| = (\lambda + \varepsilon)^{-1}, \lambda = 1/R$, $0 < \varepsilon < 1 - \lambda$, and L is an arc which joins the points $w = -1$ and $w = 1$ in the region $E(R)$, and such that $1 - 2zw + w^2 \neq 0$ for $w \in L$. The value of the radical at the point $z = w = 0$ is assumed to be equal to 1.

The proof of **(Add.1.1)** is based on the representations (1.6) and (1.7) as well as on the idea used in the proof of the well-known Hadamard theorem about the multiplication of singularities.

The relation (1.5) is given in the paper [1] of FABER but this is not pointed out by NEHARI in [1]. FABER has warned that in the case when $\mathcal{F}$ is multivalued, this relation is missing, in generally, by "passing on the other sheets" of the Riemann surface of the function $\mathcal{F}$ (G. FABER [1, p.110]). In particular, the points $w = 0, \pm 1$ may be singular for $\mathcal{F}$ even in the case when the points $z = \pm i, \pm 1$ are not singular for the function $\mathcal{A}$. It is also possible the points $w = \pm i$ to be singular for the function $\mathcal{A}$ and at the same time the point $z = 0$ not to be singular for $\mathcal{F}$. The same holds for the points $z = \infty$ and $w = 0, \infty$.

In the paper of NEHARY [1] this "disparity" of the singular points of the functions $\mathcal{F}$ and $\mathcal{A}$ is illustrated by means of a suitable example.

A generalization of (**Add.1.1**) is proposed by V.A. JACUN [1]. Denote by

$\mathcal{F}^{(\alpha,\beta)}$ the analytic function defined by the series in Jacobi polynomials

$$\sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(z). \tag{1.8}$$

Then the corresponding proposition can be stated as follows:

(Add.1.2) *Suppose that* $\operatorname{Re}\alpha > -1, \operatorname{Re}\beta > -1$ *and let* Γ *be a path which starts at a point of* $E(R) \setminus \{-1, 1\}$ *and avoids the points* $1, -1$, *and let* Γ' *and* Γ'' *be the pre-images of* Γ *by the mapping* (1.4). *Then the end point* τ *of* Γ *could be singular for the analytic function* $\mathcal{F}^{(\alpha,\beta)}$ *by the continuation of the sum of the series* (1.8) *along* Γ *if* (1.5) *holds, where* σ *is a singular point of the function* g *by its continuation along the path* $\Gamma' + \Gamma''$. *Moreover, a point* $\tau \neq \pm 1$ *is surely singular for the function* $\mathcal{F}^{(\alpha,\beta)}$ *if at least one of its pre-images by the Zhukovskii transformation is a singular point of* $\mathcal{A}$ *and at the same time it is a corner or a good accessible point of the principal Mittag-Lefler star of the function* $\mathcal{A}$. *The last assertion, in general, is not true in the case when* $\tau = \infty$ *and* $\sigma = \infty$

Mittag-Lefler's principal star $M(\mathcal{A})$ of the analytic function $\mathcal{A}$ defined by the power series (1.2) is the union of all segments $[0, r_\theta \exp i\theta)$ with $0 < r_\theta \leq \infty$ and $0 \leq \theta < 2\pi$ such that if $\theta \in [,0, 2\pi)$ and $0 < \rho < r_\theta$, then the series (1.2) is continuable along the segment $[0, \rho \exp i\theta]$ and it is not continuable along $[0, r_\theta \exp i\theta]$ if $r_\theta < \infty$. Each point $r_\theta \exp i\theta$ with $r_\theta < \infty$ is a boundary point of $M(\mathcal{A})$. It is called a corner of $M(\mathcal{A})$. A boundary point of $M(\mathcal{A})$ is called *good accessible* if there exists an open semi-disk centered at this point and lying entirely in $M(\mathcal{A})$ [L. Bieberbach, 1, **1.4**].

As a corollary of **(Add.1.2)** we can state the following proposition:

(Add.1.3) *If* $\operatorname{Re}\alpha > -1$ *and* $\operatorname{Re}\beta > -1$, *then a boundary point* τ *of the region* $E(R)$ *of convergence of the series* (1.8) *is a singular point for its sum if* (1.5) *holds, where* σ *is a singular point for the sum of the power series* (1.2).

Remark. Recall that a point z_0 on the circle $C(0; R)$ is said to be regular for the sum of the series (1.2) if there exist a disk $U(z_0; \rho)$ and a complex-valued function φ holomorphic in this disk such that $\varphi(z) = f(z)$ for each $z \in U(0; R) \cap U(z_0; \rho)$. A point on $C(0; R)$ is called singular if it is not regular. It is clear that a singular point of the series (1.2) is also a singular point for the analytic function $\mathcal{A}$ generated by this series along the radius $[0, z_0]$.

1.2 The paper [1] of Nehari is inspired by a theorem of G. Szegö [2] which clarifies the relation between singularities of the harmonic function

$$u(r, \theta) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\theta),$$

where $a_n \in \mathbb{R}, n = 0, 1, 2, \ldots$, and $\limsup_{n\to\infty} |a_n|^{1/n} = 1$, and the singularities of the power series $\sum_{n=0}^{\infty} a_n z^n$ on the unit circle.

Nehari's result is carried on series in the eigenfunctions of a suitable Sturm-Liouville problem [R.P. GILBERT and H.C. HOWARD, 1].

A recent paper [P. EBENHAFT, D. KHAVINSON and H.S. SHAPIRO, 1] presents an essentially different approach to Hehari's theorem and its Gibert's generalization (**Theorem** C of the paper just mentioned) which is based on a general principle concerning the propagation of singularities of solutions of partial differential equations.

The paper [1] of J. GUNSON and J.G. TAYLOR contains a proposition like Hadamard's multiplication theorem but for series in Legendre's polynomials. More precisely, the authors claim that a point ζ could be a singular point for the analytic function which is defined by the series $\sum\limits_{n=0}^{\infty} a_n b_n P_n(z)$ if

$$\zeta = \alpha\beta + (\alpha^2 - 1)^{1/2}(\beta^2 - 1)^{1/2}, \tag{1.9}$$

where α and β are singular points of the analytic functions defined by the series $\sum\limits_{n=0}^{\infty} a_n P_n(z)$ and $\sum\limits_{n=0}^{\infty} b_n P_n(z)$, respectively.

O.S. PARASYUK points out in his paper [1] that the relation (1.9) is a (formal) corollary of NEHARI'S result **(Add.1.1)** *via* the classical HADAMARD multiplication theorem. He points out also that a relation like (1.9) can be found in a paper of S. MANDELSTAM [1].

In another paper [2] PARASYUK formulates without full proof a proposition about the relation between singular points of the analytic functions which are defined by the series $\sum\limits_{n=0}^{\infty} a_n P_n^{(\lambda)}(z)$ and $\sum\limits_{n=0}^{\infty} \Phi(a_n) P_n^{(\lambda)}(z)$, where Φ is a complex-valued function which is holomorphic in a neighbourhood of the origin, and $\{P_n^{(\lambda)}(z)\}_{n=0}^{\infty}$ are the ultraspherical polynomials. A more general result of this kind about series of the form $\sum\limits_{n=0}^{\infty} \Phi(a_n) P_n^{(\alpha,\beta)}(z)$ is given in the above mentioned paper of JACUN [1, Theorem 4].

1.3 Suppose that $0 < A \leq \pi/2, -\pi/2 \leq B < 0, 0 < h < 1$, and denote by $G_h(A, B)$ the angular domain defined by the inequalities $B < \arg(\zeta + h) < A$. Suppose that a is a complex-valued function continuous on the closure of this domain, and holomorphic in its interior. Suppose that there exists $R \in (1, \infty)$ such that

$$a(\zeta) = O(\exp(-\zeta \log R)) \tag{1.10}$$

when $\zeta \to \infty$ in $\overline{G_h(A,B)}$.

Further, denote by $\Gamma_\theta, \theta \in \mathbb{R}$, the curve with parametric equation

$$\zeta = \cos(t\exp(-i\theta)), 0 \le t < \infty,$$

and let $D(A,B)$ be that component of the open set $\mathbb{C} \setminus \{\Gamma_{A+\pi/2} \cup \Gamma_{B-\pi/2}\}$ which contains the origin. Then, it holds the following proposition due to V.F. Cowling [1, Theorem 2.1]

(Add.1.4). *The holomorphic function defined in the region* $E(R)$ *by the series*

$$\sum_{n=0}^{\infty} a(n)P_n(z) \tag{1.11}$$

is analytically continuable in the region $D(A,B)$.

It is evident that $G_h(\pi/2, -\pi/2)$ is the half-plane defined by the inequality $\operatorname{Re}\zeta > -h$. In this case every of the curves $\Gamma_{\pi/2}$ and $\Gamma_{-\pi/2}$ coinsides with the ray $[1,\infty)$ of the real axis. Thus, we obtain the following proposition (V.F. Cowling [1, Corollary 2]):

(Add.1.5). *Suppose that* a *is a complex-valued function continuous on the closure of the half-plane* $\operatorname{Re}\zeta > -h, 0 < h < 1$, *and holomorphic in its interior. If* a *satisfies* (1.10), *then the holomorphic function defined by the series* (1.11) *is analytically continuable in the region* $\mathbb{C} \setminus [(R+R^{-1})/2, \infty)$.

1.4 A well-known fact is that the analytical continuation of a (convergent) power series centered at the origin in its principal Mittag-Leffler's star can be realized by the method of summation due also to Mittag-Leffler (G.H. Hardy [1, 8.10]).

A matrix-method for summation of series in Legendre polynomials is discussed in the paper of A. Jakimowski [1]. In fact, the author propose an effective method for analytical continuation of a holomorphic function which is defined by a (convergent) series in Legendre polynomials. To that end it is introduced the Mitag-Leffler hyperbolic star-domain of a complex-valued function which is holomorphic in a neighbourhood of the segment [-1,1]. The corresponding definition, as well as the main result, namely Theorem 2.1, are given on p. 292 of the paper just mentioned.

1.5 Series in Jacobi polynomials which define holomorphic functions with only polar singularities on the boundaries of their regions of convergence are also studied. Some results of this kind are quoted below.

(Add.1.6). [G. Boychev, 12, Theorem 1] *Suppose that the inequalities* (1.3) *hold and let the sum of the series* (1.8) *have only poles on the ellipse* $e(R)$. *If one* *of them, say z_0, has multiplicity greater that the multiplicities of the others, then there exists* $\lim_{n\to\infty}(a_n/a_{n+1}) = \omega(z_0)$.

Remark. We recall that ω is that inverse of Zhukovskii transformation for which $\omega(\infty) = \infty$.

(Add.1.7) [G. Boychev, 12, Theorem 2] *Suppose that the inequalities* (1.3) *hold. If all the singularities of the series* (1.8) *on the ellipse* $e(R)$ *are only simple poles, then*

$$a_n R^n n^{-1/2} = O(1), \quad n \to \infty. \tag{1.12}$$

The converse of the above proposition is also true:

(Add.1.8) [G. Boychev, 13, Theorem 3] *Suppose that* (1.3) *and* (1.12) *are satisfied. If all the singularities of the series* (1.8) *on the ellipse* $e(R)$ *are only poles, then they are simple.*

If (1.12) is replaced by a stronger requirement, then **(Add.1.8)** does not hold. More precisely, the following assertion is true:

(Add.1.9) [G. Boychev, 13, Theorem 5] *Suppose that* (1.3) *holds. If* $\lim_{n\to\infty} a_n R^n n^{-1/2} = 0$, *then the sum of the series* (1.8) *has no poles on the ellipse* $e(R)$.

2. Singular points and analytical continuation of series in Hermite polynomials

2.1 If the power series (1.2) has a positive radius of converegence R and if there exists a non-negative integer n_0, such that all the coefficients a_n with $n \geq n_0$ are located in an angle with vertex at the origin and opening less that π, then $w = R$ is a singular point along the radius $[0, R]$ for the analytic function $\mathcal{A}$ defined by this series.

In his *Lehrbuch der Funktionentheorie*, II, Teubner, 1927, p. 280, L. Bieberbach names the above proposition as theorem of Vivanti-Borel-Dines, but on pages 41-42 of the book L. Bieberbach [1] it is noted that the particular case when $a_n \geq 0, n \geq n_0$, is due to A. Pringsheim and that the names of Vivanti and Dines are connected with Theorem 1.8.2 on p. 40 of the same book, which is a generalization of Pringsheim's result.

Propositions like that of Pringsheim are valid for series in Hermite polynomials too.

(Add.2.1) [E. Hille, 2, Theorem 5.1] *If the series*

$$\sum_{n=0}^{\infty} a_n H_n(z) \tag{2.1}$$

has a positive ordinate of convergence τ_0*, i.e.*

$$(2.2) \qquad 0 < \tau_0 = -\limsup_{n\to\infty} (2n+1)^{-1/2} \log |(2n/e)^{n/2} a_n| < \infty,$$

and if $i^n a_n \geq 0$ *when* $n \geq n_0$*, then* $i\tau_0$ *is a singular point along the segment* $[0, i\tau_0]$ *for the analytic function* $\mathcal{H}$ *defined by the sum* f *of this series. If* $(-i)^n a_n \geq 0$ *when* $n \geq n_0$*, then the same is true for the point* $-i\tau_0$ *along the segment* $[0, -i\tau_0]$*.*

(Add.2.2) [E. Hille, 2, Theorem 5.2] *If* $a_{2n+1} = 0, n = 0, 1, 2, \dots$ *and* $(-1)^n a_{2n} \geq 0$ *for* $n \geq n_0$*, then the points* $\pm i\tau_0$ *are singular for the function* $\mathcal{H}$ *along the segment* $[0, \pm i\tau_0]$*. The same is true if* $a_{2n} = 0, n = 0, 1, 2, \dots$ *and* $(-1)^n a_{2n+1} \geq 0$ *for* $n \geq n_0$*.*

Denote $a_n = \alpha_n + i\beta_n, \alpha_n, \beta_n \in \mathbb{R}, n = 0, 1, 2, \dots$ and define for $z \in S(\tau_0)$ $= \{z \in \mathbb{C} : |\operatorname{Im} z| < \tau_0\}$:

$$f_{re}(z) = \sum_{n=0} \alpha_{2n} H_{2n}(z), \quad f_{ie}(z) = \sum_{n=0}^{\infty} \beta_{2n} H_{2n}(z),$$

$$f_{ro}(z) = \sum_{n=0}^{\infty} \alpha_{2n+1} H_{2n+1}(z), \quad f_{io}(z) = \sum_{n=0}^{\infty} \beta_{2n+1} H_{2n+1}(z).$$

The functions just defined are called components of the function f. Then the following proposition holds:

(Add.2.3) [E. Hille, 2, Theorem 5.3] *If the points* $\pm i\tau_0$ *are singular for some of the analytic functions defined by the componenets of the function* f*, then at least one of them is singular for the analytic function* $\mathcal{H}$*.*

As Hille indicates, the last proposition is analogous to a theorem due to O. Szász [1]. The following assertion is a corollary of **(Add.2.2)** and **(Add.2.3)**:

(Add.2.4) [E. Hille, 2, Theorem 5.4] *If some of the component series of* (2.1) *satisfies the conditions of* **(Add.1.1)** *and, moreover, its ordinate of convergence is equal to* τ_0*, then at least one of the points* $\pm i\tau_0$ *is singular for the analytic function* $\mathcal{H}$*.*

A particular case of the above statement is a generalization of **(Add.2.1)** which is analogous to that given by P. Dines and G. Vivanty of Pringsheim's theorem (E. Hille [2, Theorem 5.5]).

A more complicated criterion the points $\pm i\tau_0$ to be singular for an analytic function defined by series in Hermite polynomials is given in the same paper of E. Hille [2, Theorem 5.6].

2.2 In Chapter 4 of E. Hille's paper [2] is discussed the relation between the analytic continuation of the series (2.1) and that of the series

$$\sum_{n=0}^{\infty} \lambda_n a_n H_n(z), \tag{2.3}$$

where $\{\lambda_n\}_{n=0}^{\infty}$ is a sequence of complex numbers. Such a sequence is called regularity preserving factor-sequence for the series (2.1) if the analytic continuation of the series (2.1) along a path starting at the point $z = 0$ implies the same for the series (2.3).

(Add.2.5) [E. Hille, 2, Theorem 4.1] *Suppose that the complex-valued function g is defined on the set of nonnegative integers as well as that it is holomorphic in a neighbourhood of the point of infinity. If the series* (2.1) *has a positive ordinate of convergence, then the series*

$$\sum_{n=0}^{\infty} g(n) a_n H_n(z)$$

has the same ordinate of convergence and, moreover, $\{g(n)\}_{n=0}^{\infty}$ is a regularity preserving factor-sequence for the series (2.1)

(Add.2.6) [E. Hille, 2, Theorem 2.4] *If G is an entire function of growth* $[1/2; 0]$ *and if the series* (2.1) *has a positive ordinate τ_0 of convergence, then the series*

$$\sum_{n=0}^{\infty} G(n) a_n H_n(z)$$

has the ordinate of convergence at least equal to τ_0, and $\{G(n)\}_{n=0}^{\infty}$ is a regularity preserving factor-sequence for the series (2.1).

Remark. An entire function is called of growth $[\rho; \sigma], 0 < \rho < \infty, 0 \leq \sigma \leq \infty$, if it is of order ρ and type σ.

(Add.2.7) [E. Hille, 2, Theorem 4.3] *Suppose that G is an entire function of growth $[1/2; \sigma]$ with $\sigma < \infty$, and that the ordinate of convergence τ_0 of the series* (2.1) *is greater than σ. Then the ordinate of convergence of the series*

$$\sum_{n=0}^{\infty} G(2n+1) a_n H_n(z)$$

is at least equal to $\tau_0 - \sigma$. Moreover, the holomorphic function which is defined by this series is analytically continuable along every path Γ starting at $z = 0$ and such that the function f which is defined by the series (2.1) *is analytically continuable along the path Γ by means of a chain of power series having radii of convergence greater than σ.*

As E. Hille points out in his paper [1], the above propositions are analogous to theorems for Taylor and Dirichlet series due to A. Ostrowsi, G. Faber, H. Cramer, respectively.

2.3 Recall that the series (*) and (**) from **Comments and references** to Chapter IV, i.e. the series

$$C(z) = \sum_{n=0}^{\infty} a_n c_n(z) \tag{2.4}$$

and

$$S(z) = \sum_{n=0}^{\infty} a_n s_n(z), \tag{2.5}$$

have the same strip of convergence as that of the series (2.1). Moreover, there is a relation between the analytical continuations of the holomorphic functions which are defined by the series (2.1), (2.4), and (2.5).

(Add.2.8) *The function* f *which is defined by the series* (2.1) *is analytically continuable in the region* $\mathcal{C} \cap (-\mathcal{C})$*, where* $\mathcal{C}$ *is the principal Mitag-Leffler star of the function* $C(z)$ *and* $-\mathcal{C} = \{w \in \mathbb{C} : w = -z, z \in \mathcal{C}\}$.

Let us mention that the above assertion is a part of Theorem 6 in E. Hille's paper [2]. The "effective" continuaton of the series (2.1) is realized by means of the representation (6.2.9) on p. 929 of [2]. In the last paper are engaged the holomorphic functions

$$E^+(z) = \sum_{n=0}^{\infty} a_n e_n^+(z) \tag{2.6}$$

and

$$E^-(z) = \sum_{n=0}^{\infty} e_n^-(z), \tag{2.7}$$

where

$$e_n^+(z) = A_n(-i)^n \exp[i(2n+1)^{1/2}z], \; n = 0, 1, 2, \ldots,$$
$$e_n^-(z) = A_n i^n \exp[-i(2n+1)^{1/2}z], \; n = 0, 1, 2, \ldots.$$

The proposition **(Add.2.8)** remains true if we replace $\mathcal{C}$ by the principal Mittag-Leffler star of the function $S(z)$ since. as it is pointed out by E. Hille [2, p. 927], it coincides with $\mathcal{C}$. The following assertions are also proved by E. Hille:

(Add.2.9) [E. Hille, 2, Theorem 6.4] *If the function* f *defined by the series* (2.1) *is holomorphic in the circle* $U(0; R)$ (*i.e. its Taylor expansion with center at*

$z = 0$ *has radius of convergence at least equal to* R), *then the same holds for the functions* $C(z), S(z), E^{+}(z)$ *and* $E^{-}(z)$.

(Add.2.10) [E. Hille, 2, Theorem 6.5] *If* $z = z_0$ *is the singular point of* $C(z)$ *nearest to the origin, or one of the several such points, then either* z_0, *or* $-z_0$ *is a singular point of the function* f. *Conversly, each singular point of* f *is a singular point of one of the functions* $C(z)$ *and* $C(-z)$. *In particular,* f *is an entire function if and only if* $C(z)$ *is such a function.*

Suppose that G is a complex-valued function holomorphic in the angular domain $-\pi \leq -\theta_1 < \arg z < \theta_2 \leq \pi$ with positive θ_1, θ_2. Suppose that the function

$$\eta(\theta) = \limsup_{r\to\infty} r^{-1/2} \log |G(r \exp i\theta)| \tag{2.8}$$

is continuous and bounded in the interval $(-\theta_1, \theta_2)$ and, moreover, that $\eta(0) < 0$. Define the region $\Delta = \mathbb{C} \setminus \{\overline{E} \cup (-\overline{E})\}$, where $E = \{z \in \mathbb{C} : \mathrm{Im}[z \exp(i\theta/2)] \leq \eta(\theta), -\theta_1 < \theta < \theta_2)\}$. Then the following proposition holds:

(Add.2.11) [E. Hille, 3, Theorem 2] *The series*

$$\sum_{n=1}^{\infty} i^n A_n^{-1} G(n) H_n(z)$$

has the ordinate of convergence $\tau_0 = -2^{1/2}\eta(0)$, *and define a holomorphic function which is analytically contnuable in the region* Δ.

2.4 An interesting result which is given as Theorem 6.2 in E.Hille's paper [2], is the following proposition:

(Add.2.12) *There exist a Hermitian series with finite and positive ordinates of convergence which have no (finite) singular points on the boundaries of their strips of convergence. In particular, there are such series which are analytically continuable in the whole complex plane.*

Suppose that g is an entire function of order $1/2$ and normal type σ (i.e. $0 < \sigma < \infty$) which has only real and negative zeros. Then, as it is shown on p. 931 of E. Hille's paper, the series

$$\sum_{n=0}^{\infty} \frac{H_{2n}(z)}{A_{2n} g(4n+1)}$$

has ordinate of convergence σ and, moreover, it is analytically continuable in the (finite) complex plane, i.e. it defines an entire function.

Other examples of entire functions with the above property, i.e. such that their Hermite's series expansions have finite ordinates of convergence, are given also in [E. Hille, 3].

Suppose that G is a complex-valued function, holomorphic in the right half-plane and that there exist constants $A > 0$ and $B > 0$ such that

$$\limsup_{r\to\infty} r^{-1/2} \log |G(r \exp i\theta)| < A(\tan |\theta|)^{1/2} - B,$$

when $-\pi/2 < \theta < \pi/2$. Then the following assertion holds:

(Add.2.13) [E. Hille, 3. Theorem 1] *The series*

$$\sum_{n=0}^{\infty} A_n^{-1} G(n) H_n(z)$$

has ordinate of convergence not greater than $2^{-1/2}B$ *and it is analytically continuable in the whole* (*finite*) *complex plane, i.e. it defines an entire function.*

3. Singular poins and analytical continuation of series in Laguerre polynomials

3.1 Suppose that $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers such that

$$\lambda_0 = -\limsup_{n\to\infty} (2\sqrt{n})^{-1} \log |a_n| > 0. \tag{3.1}$$

Let us recall that if $\alpha + 1 \neq 0, -1, -2, \dots,$ then $\Delta(\lambda_0) = \{z \in \mathbb{C} : \operatorname{Re}(-z)^{1/2} < \lambda_0\}$ is the region of convergence of the series

$$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z). \tag{3.2}$$

Denote by f the sum of this series and let $\mathcal{L}$ be the analytic function which is defined by the holomorphic function f. The following proposition is analogous to a well-known property of the power series.

(Add.3.1) [G. Boychev, 4] *Suppose that the sequence* $\{a_n\}_{n=0}^{\infty}$ *of real numbers satisfies* (3.1) *with* $\lambda_0 < \infty$ *and let* $\{S_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ *be the partial sums of the series* (3.2). *If* $\alpha > -1$ *and* $\lim_{n\to\infty} S_n^{(\alpha)}(-\lambda_0^2) = \infty$ *(or* $-\infty$*), then* $z = -\lambda_0^2$ *is a singular point for the analytic function* $\mathcal{L}$ *which is generated by the Maclaurin series of the function* f *along the segment* $[0, -\lambda_0^2]$.

3.2 A part of the results about singular points and analytical continuation of series in Laguerre polynomials are analogous to that for series in Hermite polynomials. E.g., in the paper of L. Boyadjiev [3] the proposition **(Add.2.5)** is carried over series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ under the assumption that α is real and different from $-1, -2, -3, \dots$. Moreover, the following propositions are valid too.

(Add.3.2) [L. Boyadjiev, 5, Theorem 3] *Suppose that* $\alpha \neq -1, -2, -3, \ldots$ *is real and let the sequence* $\{a_n\}_{n=0}^{\infty}$ *satisfy* (3.1). *If the entire function* G *is of growth* $[1/2; 0]$, *then the series*

$$\sum_{n=0}^{\infty} G(n) a_n L_n^{(\alpha)}(z)$$

has region of convergence $\Delta(\lambda)$ *with* $\lambda \geq \lambda_0$ *and, moreover, the principal Mittag-Leffler star of the function which is defined by this series contains the principal Muttag-Leffler star of the function which is defined by the series* (3.2).

(Add.3.3) [L. Boyadjiev, 5, Theorem 4] *Suppose that* $\alpha \neq -1, -2, -3, \ldots$ *is real and let the sequence* $\{a_n\}_{n=0}^{\infty}$ *satisfy* (3.1) *with* $\lambda_0 < \infty$. *If the entire function* G *is of growth* $[\rho; \sigma]$ *with* $\sigma < \lambda_0$, *then the series*

$$\sum_{n=0}^{\infty} G(4n+1) a_n L_n^{(\alpha)}(z)$$

has region of convergence $\Delta(\lambda)$ *with* $\lambda \geq \lambda_0 - \sigma$ *and, moreover, the holomorphic function which is defined by this series is analytically continuable along every segment* $[0, \zeta], \zeta \in \mathbb{C}$ *such that the sum of the series* (3.1) *is continuable along it by a chain of power series with radii of convergence greater than* σ.

Suppose that the complex function G is holomorphic in the angular domain $-\pi/2 \leq -\theta_1 < \arg \theta < \theta_2 \leq \pi/2$ with $\theta_1, \theta_2 > 0$. Assume that the function (2.8) is continuous and bounded for $\theta \in (-\theta_1, \theta_2)$ and, moreover, that $\eta(0) < 0$. Define

$$V = \{z \in \mathbb{C} : \mathrm{Re}[(-z)^{1/2} \exp(i\theta/2)] \geq -\eta(\theta), 0 < \theta \leq \theta_2\},$$

and

$$\tilde{V} = \{z \in \mathbb{C} : \mathrm{Re}[(-z)^{1/2} \exp(i\theta/2)] \leq \eta(\theta), 0 < \theta \leq \theta_1\}.$$

The following proposition for series in Laguerre polynomials is analogous to **(Add.2.11)**:

(Add.3.4) [L. Boyadjiev, 11, Theorem 5] *If* $-1/2 < \alpha < 1/2$, *then the series* (3.2) *with coefficients*

$$a_n = \{\Gamma(n+\alpha+1)\}^{-1} \Gamma(n+1) G(n),\ n = 0, 1, 2, \ldots$$

has $\Delta(\lambda_0)$ *with* $\lambda_0 = -\eta(0)/2$ *as a region of convergence and, moreover, the holomorphic funtion defined by its sum, is analytically continuable in the region* $D = \mathbb{C} \setminus \{V \cup \tilde{V}\}$.

Suppose that the complex function a is continuous on the closure of the angular domain $G_h(A, B)$ which was defined in Section **1.3** and that it is holomorphic in its interior. Suppose that there exist $\lambda_0 \in (0, \infty)$ such that $a(\zeta) = O[\exp(-2\lambda_0\sqrt{\zeta})]$ when $\zeta \to \infty$ in $\overline{G_h(A, B)}$. Then the following assertion holds:

(Add.3.5) *If $\alpha \neq -1, -2, -3, \dots$ is real, then the series*

$$\sum_{n=0}^{\infty} a(n) L_n^{(\alpha)}(z)$$

converges in the region $\Delta(\lambda_0)$ and, moreover, the holomorphic function defined by its sum is analytically continuable in the angular domain $-A < \arg z < -B$.

The above proposition is proved by V.F. Cowling [1, Theorem3.1] in the case $\alpha = 0$. The generalization just given is due to L. Boyadjiev [7].

3.3 There are series in Laguerre polynomials which have regions of convergence $\Delta(\lambda_0)$ with $0 < \lambda_0 < \infty$ and such that every (finite) point of the boundary of $\Delta(\lambda_0)$ is regular (i.e. not singular) for the holomorphic functions which they define. Such examples are analogous to those proposed by E. Hille and considered in the previous Section.

(Add.3.6) [L. Boadjiev, 11, Theorem 4] *Suppose that the entire function g of growth $[1/2; \sigma], 0 < \sigma < \infty$ has only real and negative zeros. If $-1/2 < \alpha < 1/2$, then the series*

$$\sum_{n=0}^{\infty} (-1)^n \Gamma(n+1) \{\Gamma(n+\alpha+1) g(4n+1)\}^{-1} L_n^{(\alpha)}(z)$$

has $\Delta(\sigma)$ as its region of convergence and, moreover, its sum is analytically continuable in the whole (finite) complex plane, i.e. it defines an entire function.

(Add.3.7) [L. Boyadjiev, 1,Theorem 5] *Suppose that the complex-valued function G is holomorphic in the right half-plane. If*

$$\limsup_{r\to\infty} r^{-1/2} \log |G(r \exp i\theta)| = A(\tan|\theta|)^{1/2} - 2\lambda_0, \ -\pi/2 < \theta < \pi/2,$$

where $A > 0$ and $0 < \lambda_0 < \infty$, and $-1/2 < \alpha < 1/2$, then the series

$$\sum_{n=1}^{\infty} (-i)^n \Gamma(n+1) \{\Gamma(n+\alpha+1)\}^{-1} G(n) L_n^{(\alpha)}(z)$$

has $\Delta(\lambda_0)$ as its region of convergence. Moreover, its sum is analytically continuable in the whole (finite) complex plane.

3.4 It seems that the analytical continuation of series in Laguerre's polynomials by means of Borel's and Mittag-Leffler's methods of summability is not yet studied systematically. Boas and Buck gave sufficient conditions for a Laguerre expansion of an entire function f of exponential type to be Mittag-Leffler-summable. This is really the fact when the conjugate indicator diagram of f avoids the ray $(1, \infty)$ of the real line [Boas and Buck, 1, p.40,(X)].

In [G. Boychev and P. Rusev, 1] it is proved that the series in the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ which represents the function [V, (3.18)] in the region $\Delta(\lambda_0), 0 < \lambda_0 < \infty$, is B'-summable (i.e. it is summable by means of Borel's integral method [G. Hardy, 1, **8.5**]) in the region $\Delta(\lambda_0\sqrt{2})\setminus(-2\lambda_0^2, -\lambda_0^2]$ provided that $\alpha > -1/2$.

4. Gap theorems and overconvergence

4.1 From **(Add.1.3)** it follows that if the power series (1.2) has radius of convergence $R \in (0, \infty)$ and, moreover, every point of the circle $C(0; R)$ is singular for its sum, i.e. the disk $U(0; R)$ is the domain of existence of the analytic function $\mathcal{A}$ which is defined by this series, then each point of the boundary $e(R)$ of the region of convergence $E(R)$ of the series (1.8) is singular for its sum, i.e. $E(R)$ is the domain of existence of the analytic function $\mathcal{F}^{(\alpha,\beta)}$ defined by the series (1.8). In particular, it follows that for series of the kind (1.8) it holds a proposition which is analogous to the classical gap theorems, due to J. Hadamard and E. Fabry [L. Bieberbach, 1, Theorem 1.8.5 and Theorem 2.2.1], about the analytical uncontinuability of power series of the kind

$$\sum_{k=0}^{\infty} a_k w^{n_k}, \tag{4.1}$$

where $\{n_k\}_{k=0}^{\infty}$ is an increasing sequence of non-negative integers. As their corollaries we can state the following propositions:

(Add.4.1) *Suppose that $\{n_k\}_{k=0}^{\infty}$ is a sequence of non-negative integers such that*

$$n_{k+1} > (1+\theta)n_k, \; k = 0, 1, 2, \dots, \tag{4.2}$$

where $\theta > 0$ and let $\{a_k\}_{k=0}^{\infty}$ be a sequence of complex numbers such that

$$1 < R = (\limsup_{n\to\infty} |a_k|^{1/n_k})^{-1} < \infty. \tag{4.3}$$

If $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\beta > -1$, then $E(R)$ is the domain of existence of the analytic function which is defined by the sum of the series

$$\sum_{k=0}^{\infty} a_k P_{n_k}^{(\alpha,\beta)}(z). \tag{4.4}$$

(Add.4.2) *The proposition* **(Add.4.1)** *holds if*

$$\lim_{k\to\infty} \frac{k}{n_k} = 0. \tag{4.5}$$

Remarks: (1) Condition (4.5) is much more weaker than (4.2). The latter can be expressed also in the following way:

$$\liminf_{k\to\infty} \{n_{k+1} - n_k\} > 0.$$

(2) E. Borel has changed the above condition by the following:

$$\liminf_{k\to\infty} \frac{n_{k+1} - n_k}{\sqrt{n_k}} > 0.$$

(3) G. Faber has proved Hadamard's gap theorem under each of the assumptions

$$\sum_{k=1}^{\infty} n_k^{-\sigma} < \infty,\ 0 < \sigma < 1,$$

and

$$\liminf_{k\to\infty} \frac{n_{k+1} - n_k}{n_k^{\sigma}},\ \sigma > 0.$$

(4) E. Fabry has proved that **(Add.4.2)** holds when

$$\lim_{k\to\infty} (n_{k+1} - n_k) = \infty.$$

The last requirement implies (4.5) but, in general, the converse is not true. (Example: $n_{2k} = k^2, n_{2k+1} = k^2 + 1$).

(5) The above propositions along with corresponding references can be found in [L. Bieberbach, 1, **2.2**].

4.2 Series in Hermite polynomials (more exactly in Hermite functions) with gaps have been studied by E. Hille [2]. In order to state some of his results we need some definitions and notations.

Suppose that $\{\lambda_k\}_{k=1}^{\infty}$ is a monotonically increasing sequence of real and positive numbers. Denote by $N(t)(0 < t < \infty)$ the number of λ's which are $\leq t$, and define $\omega(t) = t^{1/2}(N(t))^{-1}$, and $\omega^*(t) = \inf_{u\geq t} \omega(u)$.

In Chapter 5 of his paper [2] E. Hille formulates the following hypothesis:

Hypothesis A: $\omega(t) \to \infty$ when $t \to \infty$.

Hypothesis B: $\omega(t)(\log t)^{-1} \to \infty$ when $t \to \infty$.

Hypothesis C: Hypothesis A holds and, moreover, $\lambda_{k+1} - \lambda_k \geq c\lambda_k^{1/2}$ with some $c > 0$.

Hypothesis D: Hypothesis A holds and, moreover, there exists $\mu \geq 0$ such that for every $\varepsilon > 0$,

$$\lambda_{k+1} - \lambda_k \geq \lambda_k^{1/2} \exp[-(\mu + \varepsilon)\omega^*(k)]$$

when $k \geq k_0(\varepsilon)$.

(Add.4.3) [E. Hille, 2, Theorem 5.7] *Suppose that* $\{n_k\}_{k=1}^{\infty}$ *is a sequence of positive integers which satisfies either of Hypotheses B,C or D* (*with* $\mu = 0$). *If the series*

$$\sum_{k=1}^{\infty} a_k H_{n_k}(z) \tag{4.6}$$

has positive ordinate of convergence τ_0, *then the strip* $S(\tau_0)$ *is the domain of existence of the analytic function defined by its sum.*

Remarks: (1) In the case $\lambda_k = n_k,\ k = 1, 2, 3 \ldots$, *Hypothesis A* is equivalent to $k^{-2}n_k \to \infty$ when $k \to \infty$.

(2) As E. Hille has pointed out [2, p.932, Corollary], the condition $n_k = O(k^2)$ is not sufficient to ensure the non-continuability of the series (4.6).

(Add.4.4) [E. Hille, 2, Theorem 5.8] *Suppose that* $\{n_k\}_{k=1}^{\infty}$ *is an increasing sequence of positive integers which satisfies the Hypothesis D. Then the region of existence of the analytic function which is defined by the series* (4.6) *is contained in the strip* $S(\tau_0 + \mu)$

Using **(Add.4.3)** E. Hille has got the following result [2, Theorem 5.9]:

(Add.4.5) *If the series* (2.1) *has a finite and positive ordinate of convergence* τ_0, *then the set of series*

$$\sum_{n=0}^{\infty} \varepsilon_n a_n H_n(z),\ \varepsilon = \pm,\ n = 0, 1, 2, \ldots,$$

which are analytically non-continuable outside the strip $S(\tau_0)$ (*i.e. for which each point on the lines* $\operatorname{Im} z = \pm\tau_0$ *is singular*), *is non-denumerable.*

This assertion is analogous to a classical theorem for power series formulated at first by P. Fatou and proved later by A. Hurwitz and G. Pólya.

4.3 The above theorems of E. Hille are transfered for series in Laguerre polynomials by L. Boyadjiev [11, Theorem 5]. If the sequence $\{n_k\}_{k=1}^{\infty}$ satisfies suitable modifications of the *Hypotheses* B,C,D and, moreover,

$$0 < -\limsup (2\sqrt{n_k})^{-1} \log |a_k| = \lambda_0 < \infty, \tag{4.7}$$

then $\Delta(\lambda_0)$ is the naturale domain of existence for the analytic function which is defined by the series

$$\sum_{k=0}^{\infty} a_k L_{n_k}^{(\alpha)}(z),\ -1/2 < \alpha < 1/2. \tag{4.8}$$

Propositions like **(Add.4.3), (Add.4.4)** and **(Add.4.5)** are also established [L. Boyadjiev, 11, Theorems 6,7,8].

4.4 A. Ostrowski discovered a remarkable property of the power series with gaps and named it "Überkonvergenz" (overconvergence) [1].

We say that the power series (1.2) with positive radius of convergence R has Hadamard's gaps if there exist two increasing sequences $\{p_k\}_{k=0}^{\infty}$ and $\{q_k\}_{k=0}^{\infty}$ of positive integers and a real number $\theta > 0$ such that

$$q_k > (1+\theta)p_k,\ k = 0, 1, 2, \dots \tag{4.9}$$

and, moreover, $a_n = 0$ if $p_k < n \leq q_k,\ k = 0, 1, 2, \dots$. Then, as Ostrowski has proved, the subsequece of partial sums

$$\left\{\sum_{n=0}^{p_k} a_n w^n\right\}_{k=0}^{\infty} \tag{4.10}$$

is uniformly converegent in a neighbourhood of every point of the circle $C(0; R)$ which us regular for the series (1.2). To this end he uses the classical Hadamard's three-cicle theorem. More precisely, he applies it to the functions $\{\varphi(w) - s_{p_k}(w)\}_{k=0}^{\infty}$, where $\varphi(w)$ is the sum of the series (4.10) and $\{s_\nu(w)\}_{\nu=0}^{\infty}$ is the sequence of its partial sums.

Hadamard's gap theorem is a corollary of Ostrowski's theorem. Indeed, if we define $p_k = n_k, q_k = n_{k+1},\ k = 0, 1, 2, \dots$, then (4.2) is satisfied and, moreover, (4.10) is just the sequence of the partial sums of the series (4.1).

A. Ostrowski has shown also that the overconvergence of a power series is closely related with the existence of gaps. More precisely, he has proved that if a sequence of partial sums of the series (1.2) with radius of convergence $R \in (0, \infty)$ is uniformly convergent in a neighbourhood of a point of the circle $C(0; R)$ which is regular for its sum, then this series can be represented as a sum of a series with Hadamard's gaps and a series with a radius of convergence, greater than R.

4.5 Ostrowski's overconvergence theorem has been proved for series in other systems of holomorphic functions and, in patricular, for series in denumerable systems of polynomials. A result of this kind is stated e.g. in a paper of S.Ya. Al'per [1].

Suppose that K is a nonempty compact subset of the complex plane with a simply connected complement G. If K contains more than one point, then denote by r the conformal radius of the region G with respect to the point of infinity and let $r = \infty$ when K is a single point. If $r < \infty$, then we denote by $w = \varphi(z)$ the conformal mapping of G on the region $\{w \in \mathbb{C} : |w| > r^{-1} \cup \{\infty\}\}$ for which $\varphi(\infty) = \infty$ and $\varphi'(\infty) = 1$. In the case $r = \infty$, i.e. when $K = \{z_0\}$ with $z_0 \in \mathbb{C}$, let $\varphi(z) = z - z_0$. Then we have the following proposition:

(Add.4.8) *Suppose that* $\{p_n(z)\}_{n=0}^{\infty}$, $\deg p_n = n, n = 0, 1, 2, \ldots$ *is a system of polynomials with the property that for every* $R > r^{-1}$, *and* $\varepsilon > 0$ *there exists* $M(R, \varepsilon) < \infty$ *such that the inequality* $|p_n(z)| \leq M(R, \varepsilon)(R + \varepsilon)^n$ *holds for* $n = 0, 1, 2, \ldots,$ *and* $z \in C_R$, *where* C_R *is the pre-image of the circle* $C(R; 0)$ *by the mapping* φ. *Then an overconvergence theorem of Ostrowski's type holds for series of the kind* $\sum_{n=0}^{\infty} a_n p_n(z)$ *provided* $0 < \limsup_{n\to\infty} |a_n|^{1/n} = \rho^{-1} < r^{-1}$.

Remark. The proof can be carried out in the lines of that of the classical Ostrowski's theorem, i.e. by using Hadamard's three circles theorem.

As a corollary of the above proposition as well as of the asymptotic formula [Chapter III, (1.9)] for Jacobi polynomial we obtain that a theorem of Ostrowski's type holds for series in these polynomials:

(Add.4.9) *Suppose that* $\alpha+1, \beta+1$ *and* $\alpha+\beta+2$ *are not equal to* $0, -1, -2, \ldots$. *If the series* (1.8), *whose coefficients satisfy* (1.3), *has* Hadamard's *gaps, then there exists a subsequence of its partial sums which is uniformly convergent in a neighbourhood of every point of the ellipse* $e(R)$ *provided that it is regular for its sum.*

4.6 Ostrowski type overconvergence theorem holds also for series in Laguerre polynomials:

(Add.4.10) [P. Rusev, 1, Theorem 2] *Suppose that* α *is an arbitrary complex nuber and that* $\{a_n\}_{n=0}^{\infty}$ *is a sequence of complex numbers which satisfies* (3.1) *with* $0 < \lambda_0 < \infty$. *Let there exist two sequences* $\{p_k\}_{k=0}^{\infty}$, *and* $\{q_k\}_{k=0}^{\infty}$ *of positive integers such that* (4.10) *holds and, moreover,* $a_n = 0$ *when* $p_k < n < q_k, k = 0, 1, 2 \ldots$. *Then the sequence of partial sums*

$$\left\{\sum_{n=0}^{p_k} a_n L_n^{(\alpha)}(z)\right\}_{k=0}^{\infty}$$

is uniformly convergent in a neighbourhood of every regular point $z \in p(\lambda_0) = \partial\Delta(\lambda_0)$ *for the holomorphic function, defined by the series* (3.1).

Ostrowski's idea is followed in the proof of the above proposition but, instead of the series (3.1), we consider the series

$$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z^2). \tag{4.11}$$

Notice hat if (3.1) holds, then the region of convergence of the series (4.11) is the vertical strip $\zeta \in \mathbb{C} : |\operatorname{Re}\zeta| < \lambda_0$.

In a paper of Mu Lehua [1] is given a proof of **(Add.4.10)** for the series (3.2) provided $-\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n| = 1$ and that α is real and greater than -1. The author applies Hadamard's three-circle theorem directly to the functions $\{f(z) - S_{p_k}(z)\}_{k=0}^{\infty}$, where f is the sum of the series (3.2) and $\{S_\nu(z)\}_{\nu=0}^{\infty}$ is the sequence of its partial sums.

As a corollary of **(Add.4.10)** we can formulate a theorem of Hadamard's type for series in Laguerre polynomials.

(Add.4.11) [P. Rusev, 1, Theorem 3] *Suppose that α is an arbitrary complex number and that $\{n_k\}_{k=0}^{\infty}$ is a sequence of positive integers which satisfies* (4.2) *with $\theta > 0$. If $\{a_k\}_{k=0}^{\infty}$ are complex numbers such that the inequalities* (4.7) *hold, then the series* (4.8) *defines an analytic function for which the region $\Delta(\lambda_0)$ is its domain of existence.*

As it was just mentioned, the proposition **(Add.4.10)** is proved by using Hadamard's three circles theorem. The same techniques can be applied to series in Hermite polynomials. Thus, we obtain the following proposition:

(Add.4.12) *Suppose that the series* (2.1) *has ordinate of convergence $\tau_0 \in (0,\infty)$ and that there exist two increasing sequences $\{p_k\}_{k=0}^{\infty}$ and $\{q_k\}_{k=0}^{\infty}$ of positive integers which satisfy* (4.9)*. If $a_n = 0$ when $p_k < n < q_k,\ k = 0,1,2,\dots,$ then the sequence of partial sums*

$$\left\{\sum_{n=0}^{p_k} a_n H_n(z)\right\}_{k=0}^{\infty}$$

of the series (2.1) *is uniformly convergent in a neighbourhood of each boundary point of the strip $S(\tau_0)$ provided it is regular for the holomorphic function, defined by the series* (2.1).

Comments and references

In the paper of Jutta Faldey [1] it is shown that gap theorems of Fabry's type for lacunary series in the classical Jacobis, Laguerre and Hemite polynomials can be derived in a unified approach bythe use of certain differential operators of infinite order.

Let us mention also that in [J. Faldey, 2] are proved Ostrowski's theorem on overconvergence, Hadamard's gap theorem, and Fatou-Hurwitz-Pólya's theorem for series in Rakhmanov's type polynomials [E.A. Rakhmanov, 1].

If $\limsup_{n\to\infty}|a_n|^{1/n} = 1$, then the series (1.1) diverges at each point outside the segment $[1, 1]$. Under the additional asumption that $|a_n| = O(n^p), n = 1,2,3\dots,$ for some positive integer p, G.G. Walter [2] proves that this series "converges" in $(-1,1)$ to a distribution g and that the analytic representation $\hat{g}$ of g is given by the series

$$\sum_{n=0}^{\infty} i\pi^{-1} a_n Q_n(z),\ z \in \mathbb{C} \setminus [-1,1],$$

where $\{Q_n(z)\}_{n=0}^{\infty}$ are the Legendre associated functions. Moreover, the function g has a singular point $z_0 = (w_0 + w_0^{-1})/2$ in $(-1,1)$ if and only if the function which is defined by the power series (1.2) has $w_0 \neq \pm 1$ and $\overline{w_0}$ as singular points on the unit circle [G.G. Walter, 2, 3. The main theorem].

At the end of [G.G. Walter, 2] the author points out: "Our result can be extended *quid pro quo* to Jacobi polynomials $P_n^{(\alpha,\beta)}(z), \alpha, \beta > -1$" and afterwards he gives a "sketch" of the proof.

The validity of G.G. Walter's proposition just mentioned is confirmed later by A.I. Zayed [1] in the case $\alpha = \beta$, i.e. for expansions in ultraspherical polynomials.

Suppose that $d_1, d_2, d_3, \dots, d_k,\ k \geq 1$, are distinct non-zero complex numbers and that each of the complex numbers $a_n,\ n - 0, 1, 2, \dots$, be equal to one of d's. Then, as it is proved by G. Szegö [3], either the sum of the power series (1.2) is a rational function of the form $P(z)(1 - z^k)^{-1}$, where P is a polynomial, or it is not analytically continuable outside the unit disk. Moreover, the first case occurs if and only if there exists an integer ν such that the sequence $\{a_n\}_{n=\nu}^{\infty}$ is periodic.

In [P. Rusev, 30] it is shown that a theorem of Szegö's type holds for series in polynomials which are orthogonal on the segment [-1,1] with respect to a suitable weight function. As a corollary we can assert that if $\alpha, \beta > -1$ and $a_n = \lambda_n R^{-n},\ R > 1, n = 0, 1, 2, \dots$, where each $\lambda_n,\ n = 0, 1, 2, \dots$, is equal to one of the complex numbers $d_1, d_2, d_3, \dots, d_k$, then the function defined by the series (1.8) is analytically non-continuable outside the ellipse $e(R)$ provided that the sequence $\{\lambda_n\}_{n=\nu}^{\infty}$ is not periodic for each $\nu = 0, 1, 2, \dots$.

Remark. A theorem of Szegö's type holds even for series in systems of polynomials with the property described in **(Add.4.8)**. It means that the above corollary concerning Jacobi's polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{\infty}$ holds provided α, β are arbitrary complex numbers.

A remarkable generalization of Fabry's theorem [L. Bieberbach, 1, Theorem 2.2.1] is due to G. Pólya [1]. He has proved that if the power series (4.1) has positive and finite radius of convergence R and, moreover, if there exists

$$(*) \qquad \lim_{k\to\infty} \frac{k}{n_k} = \delta,$$

then each closed arc $C_\delta(R)$ of the circle $C(0;R)$ with angular measure $2\pi\delta$ contains at least one singular point of the analytic function defined by this series.

Denote by $e_\delta(R)$ the image of the arc $C_\delta(R)$ by the Zhukovskii transformation. As a corollary of **(Add.1.3)** we can assert that if (1.3) and (*) hold, then the series (4.4) has at least one singular point on each arc of the kind $e_\delta(R)$.

Theorems of PÓLYA'S type are obtained by A.F. LEONT'EV for series in Jacobi, Hermites and Laguerre polynomials as corollaries of more general reuslts about the singularities of holomorphic functions which are defined as limits of sequences of linear combinations of solutions of linear second order differential equations. The corresponding propositions can be found in [A.F. LEONT'EV, 1, Theorems 3.2.15, 3.2.16, 3.2.17].

It seems that most of the results about singular points as well as about analytical continuation of series in the classical orthogonal polynomials can be carried over series in the corresponding systems of associated functions. This is realy the case when e.g theorems of Vivanti-Dines' type are in question [G. BOYCHEV, 4, Theorems 4,5]. It is not difficult also to prove gap theorems of Ostrowski's type for series in Laguerre's as well as in Hermite associated functions.

Suppose that the series (3.2) converges in the whole complex plane, i.e. its sum is an entire function. If, in addition, this function is of exponential type, then, in general, there exists a relation between the singularities of its Borel transform and that of the power series (1.2). More precisely, if $\alpha > -1$ and if the series (3.2) represents an entire function f of exponential type less that one, then a point γ is singular for its Borel transform $B_f(w)$ if and only if the point $1 - \gamma^{-1}$ is singular for the power series (1.2). This assertion which can be regarded as a theorem of Faber-Nehari's type for series in Laguerre's polynomials is proved by A.J. ZAYED [2] for $\alpha = 0$ and by L. BOYADJIEV [1] for arbitrary $\alpha > -1$.

Appendix

A SHORT SURVEY ON SPECIALL FUNCTIONS

1. Gamma-function

1.1 The points $0, -1, -2, \dots$ are the only zeros of the entire function

$$F(z) = \exp(Cz)z \prod_{n=1}^{\infty} \Big(1 + \frac{z}{n}\Big) \exp\Big(-\frac{z}{n}\Big) \tag{1.1}$$

and, moreover, all they are simple. The constant C is uniquely determined by the requrement $F(1) = 1$ which leads to the equality

$$C = \lim_{n\to\infty} \Big\{ \sum_{k=1}^{n} \frac{1}{k} - \log(n+1) \Big\}. \tag{1.2}$$

Usually C is called Euler-Masceroni's constant.

By definition,

$$\Gamma(z) = \frac{1}{F(z)},\ z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}. \tag{1.3}$$

Hence, the Γ-function is a meromorphic function with only simple poles at the points $0, -1, -2, \dots$. Moreover, $\Gamma(z) \neq 0$ for $z \in \mathbb{C}$, i.e. the Γ-function has no zeros. In particular, $\Gamma(1) = 1$.

1.2 The most popular integral representation of Γ-function is that in the half-plane $\operatorname{Re} z > 0$, i.e.

$$\Gamma(z) = \int_0^{\infty} t^{z-1} \exp(-t)\, dt,\ \operatorname{Re} z > 0. \tag{1.4}$$

From (1.4) it follows that

$$\Gamma(z) = \int_0^{1} t^{z-1} \exp(-t)\, dt + g(z),$$

where

$$g(z) = \int_1^{\infty} t^{z-1} \exp(-t)\, dt$$

is an entire function. Using the expansion of $\exp(-t)$ in a power series centered at the origin we easily find that if $\operatorname{Re} z > 0$, then

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z+n} + g(z). \tag{1.5}$$

By the identity theorem the above representation holds in the region $G = \mathbb{C} \setminus \{0, -1, -2, \dots\}$.

1.3 Integrating by parts in (1.4), we obtain the relation

$$\Gamma(z+1) = z\Gamma(z) \tag{1.6}$$

provided $\operatorname{Re} z > 0$. Again, the identity theorem yields that it holds in the whole region G. In particular, since $\Gamma(1) = 1$, we have

$$\Gamma(n+1) = n!,\ n = 0, 1, 2, \dots. \tag{1.7}$$

Using the functional equation (1.6) and the defining equality (1.3), we find that for $z \in \mathbb{C} \setminus \mathbb{Z}$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}\cdot \tag{1.8}$$

Since $\Gamma(1/2) > 0$, it follows that

$$\Gamma(1/2) = \sqrt{\pi}. \tag{1.9}$$

1.4 Define the complex-valued function γ in the region $\mathbb{C} \setminus (-\infty, 0]$ by the equality

$$\Gamma(z) = \sqrt{2\pi} z^{z-1/2} \exp(-z)\{1 + \gamma(z)\}. \tag{1.10}$$

Then, whatever $0 < \delta < \pi$ and $\rho \in \mathbb{R}^+$ be, the function $z\gamma(z)$ is bounded in the region $D(\delta, \rho)$ defined by the inequalities $|\arg z| < \pi - \delta$ and $|z| > \rho$. In particular, $\gamma(z) = O(z^{-1})$ when $z \to \infty$ and $|\arg z| < \pi - \delta$.

The representation (1.10) is called Stirling's formula for Γ-function. In particular,

$$n! = \sqrt{2\pi} n^{n+1/2} \exp(-n)\{1 + \gamma_n\}, \tag{1.11}$$

where $\gamma_n = O(n^{-1})$ when $n \to \infty$.

It can be proved that if α and β are arbitrary complex numbers, then there exist constants $\{\gamma_k(\alpha,\beta)\}_{k=0}^{\infty}(\gamma_0(\alpha,\beta) = 1)$ and a sequence of complex functions $\{\gamma_\nu^{(\alpha,\beta)}(z)\}_{\nu=1}^{\infty}$ which are holomorphic in a region of the kind $E(\alpha,\beta) = \{z \in \mathbb{C} : d(z, (-\infty, 0]) > \rho_0 = \rho_0(\alpha,\beta) > 0\}$ ($d(z, (-\infty, 0])$ is the distance of the point z to the ray $(-\infty, 0]$), and such that for $\nu = 1, 2, 3 \dots$

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\Big\{\sum_{k=0}^{\nu-1} \gamma_k(\alpha,\beta) z^{-k} + \gamma_\nu^{(\alpha,\beta)}(z)\Big\},\ z \in E(\alpha,\beta). \tag{1.12}$$

Moreover, if $\delta \in (0, \pi)$, then $\gamma_\nu^{(\alpha,\beta)}(z) = O(z^{-\nu})$ when $z \to \infty$ in $E(\alpha, \beta)$ and $|\arg z| < \pi - \delta$. For example,

$$\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} = n^{\alpha-\beta}\Big\{\sum_{k=0}^{\nu-1} \gamma_k(\alpha,\beta) n^{-k} + \gamma_{\nu,n}^{(\alpha,\beta)} n^{-\nu}\Big\}, \tag{1.13}$$

where $\gamma_{\nu,n}^{(\alpha,\beta)} = O(1)$ when $n \to \infty$, and, in particular,

$$\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} = n^{\alpha-\beta}\Big\{1 - \frac{(\alpha-\beta)(\alpha+\beta-1)}{2n} + O\Big(\frac{1}{n^2}\Big)\Big\}, \ n \to \infty. \tag{1.14}$$

1.5 The holomorphic function $B(z, w)$, defined by the equality

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\, dt, \tag{1.15}$$

provided that $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$, is called beta-function (B-function). It can be expressed by means of Γ-function, i.e.,

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{1.16}$$

2. Bessel functions

2.1 The equation

$$z^2 w'' + z w' + (z^2 - \alpha^2) w = 0 \tag{2.1}$$

is called the Bessel differential equation. Its (analytic) solutions are called the cylinder functions. Bessel functions are special kind of cylinder functions.

The power series

$$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu \left(\frac{z}{2}\right)^{2\nu}}{\nu!\Gamma(\nu+\alpha+1)} = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu F(\nu+\alpha+1)}{\nu!}\Big(\frac{z}{2}\Big)^{2\nu}, \ \alpha \in \mathbb{C}$$

is absolutely convergent in the whole complex plane. This can be proved e.g. by using the well-known geometric test. Moreover, it is easy to verify that the entire function, defined by this series, satisfies the differential equation

$$z^2 w'' + (2\alpha+1) z w' + z^2 w = 0.$$

Therefore, the complex-valued function, defined in the region $\mathbb{C} \setminus (-\infty, 0]$ by the equality

$$J_\alpha(z) = \Big(\frac{z}{2}\Big)^\alpha \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \left(\frac{z}{2}\right)^{2\nu}}{\nu!\Gamma(\nu+\alpha+1)}, \tag{2.2}$$

is a holomorphic solution of the equation (2.1) in this region. It is called Bessel function of the first kind with index (or parameter) α.

If α is not an integer, then $J_\alpha(z)$ and $J_{-\alpha}(z)$ are linearly independent solutions of the equation (2.1) in the region $\mathbb{C} \setminus (-\infty, 0]$.

If $\alpha = -n$ is a negative integer, then the first n terms of the series in the right-hand side of (2.2) vanish, and a simple calculation leads to the relation

$$J_{-n}(z) = (-1)^n J_n(z), \; n = 1, 2, 3 \dots. \tag{2.3}$$

2.2 After replacing z by iz in (2.1) we obtain the equation

$$z^2 w'' + z w' - (z^2 + \alpha^2) w = 0. \tag{2.4}$$

It is called the modified Bessel differential equation. Evidently, the function

$$I_\alpha(z) = \exp(-i\alpha\pi/2) J_\alpha(iz) = \left(\frac{z}{2}\right)^\alpha \sum_{\nu=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2\nu}}{\nu!\Gamma(\nu+\alpha+1)} \tag{2.5}$$

(as well as the function $I_{-\alpha}(z)$) is a solution of the equation (2.4) in the region $\mathbb{C} \setminus (-\infty, 0]$. It is called modified Bessel function of the first kind with index α. If α is not an integer, then I_α and $I_{-\alpha}$ are linearly independent in the region $\mathbb{C} \setminus (-\infty, 0]$, but $I_{-n}(z) = I_n(z)$ for $n = 1, 2, 3 \dots$.

If α is not an integer, then the function

$$K_\alpha(z) = \frac{\pi}{2 \sin \alpha\pi} \{I_{-\alpha}(z) - I_\alpha(z)\} \tag{2.6}$$

which is also a solution of (2.4) in the region $\mathbb{C} \setminus (-\infty,]$ is called modified Bessel function of the third kind with index α. Let us note that $K_{-\alpha}(z) = K_\alpha(z)$.

By definition $K_n(z) = \lim_{\alpha \to n} K_\alpha(z)$, $n \in \mathbb{Z}$, i.e. the relation $K_{-n}(z) = K_n(z)$ is still valid when n is an arbitrary integer.

It can be proved that the representation

$$K_n(z) = (-1)^{n+1} I_n(z) \log(z/2) + \psi_n(z), \; n = 0, 1, 2, \dots \tag{2.7}$$

holds in the region $\mathbb{C} \setminus (-\infty, 0]$, where ψ_n is an entire function.

2.3 There exists numerious relations between Bessel functions both of the same and different kind. We need only some of the so called recurrence formulae:

$$J_{\alpha-1}(z) + J_{\alpha+1}(z) = 2\alpha z^{-1} J_\alpha(z), \tag{2.8}$$

$$I_{\alpha-1}(z) - I_{\alpha+1}(z) = 2\alpha z^{-1} I_\alpha(z), \tag{2.9}$$

$$K_{\alpha-1}(z) - K_{\alpha+1}(z) = -2\alpha z^{-1} K_\alpha(z). \tag{2.10}$$

If $\alpha = n + 1/2,\ n \in \mathbb{Z}$, then all the Bessel functions become elementary or can be expressed as linear combinations of such functions. For instance, we have

$$(a)\ J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z;\ (b)\ J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z; \tag{2.11}$$

$$(a)\ I_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cosh z;\ (b)\ I_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sinh z; \tag{2.12}$$

$$K_{-1/2}(z) = K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} \exp(-z). \tag{2.13}$$

The first two of the above formulae follow from the defining equality (2.2). The validity of (2.12) (a),(b) follows from the definition of the function I_α by (2.5). The last two equalities follow immediately from (2.6).

2.4 If $\operatorname{Re}\alpha > -1/2$, then for $z \in \mathbb{C} \setminus (-\infty, 0]$ the equality

$$\Gamma(\alpha + 1/2) J_\alpha(z) = \frac{2}{\sqrt{\pi}} \Big(\frac{z}{2}\Big)^\alpha \int_0^1 (1 - t^2)^{\alpha - 1/2} \cos zt\, dt \tag{2.14}$$

holds. It is called Poisson's integral representation of the function J_α.

In order to give here another integral representation of the function J_α, we denote by $L(\delta, \rho)$ the boundary of the region $D(\delta, \rho)$ defined in the previous section. Suppose that $\delta \in (0, \pi/2)$ and that $L(\delta, \rho)$ is counterclockwise oriented, then

$$J_\alpha(z) = \frac{1}{2\pi i} \Big(\frac{z}{2}\Big)^\alpha \int_{L(\delta,\rho)} \zeta^{-\alpha-1} \exp(\zeta - z^2/4\zeta)\, d\zeta. \tag{2.15}$$

for $z \in \mathbb{C} \setminus (-\infty, 0]$.

If $\operatorname{Re}\alpha > -1/2$, then the integral representation

$$\Gamma(\alpha + 1/2) K_\alpha(z) = \sqrt{\pi} \Big(\frac{z}{2}\Big)^\alpha \int_1^\infty (t^2 - 1)^{\alpha - 1/2} \exp(-zt)\, dt \tag{2.16}$$

holds for the modified Bessel function K_α in the half-plane $\operatorname{Re} z > 0$.

2.5 Denote for $\alpha \in \mathbb{C}$ and $k = 0, 1, 2, \ldots$

$$(\alpha, k) = \frac{1}{k! 2^{2k}} (4\alpha^2 - 1^2)(4\alpha^2 - 3^2) \ldots (4\alpha^2 - (2k-1)^2).$$

If α is real, then the representation ($\nu = 1, 2, 3, \dots$)

$$(2.17)\qquad J_\alpha(z) = \sqrt{\frac{2}{\pi z}}\Big\{\cos(z - \alpha\pi/2 - \pi/4)\Big[\sum_{k=0}^{\nu-1}(-1)^k(\alpha, 2k)(2z)^{-2k} + \xi_\nu(z)\Big]$$

$$- \sin(z - \alpha\pi/2 - \pi/4)\Big[\sum_{k=0}^{\nu-1}(-1)^k(\alpha, 2k+1)(2z)^{-2k-1} + \eta_\nu(z)\Big]\Big\}$$

holds in the region $\mathbb{C}\backslash(-\infty, 0]$, where ξ_ν and η_ν are complex-valued functions which are holomorphic in this region. Moreover, if $\delta \in (0, \pi)$, then $\xi_\nu(z) = O(z^{-2\nu})$ and $\eta_\nu = O(z^{-2\nu-1})$ when $z \to \infty$ and $|\arg z| \le \pi - \delta$.

For the function $K_\alpha(z)$ it holds the asymptotic formula

$$(2.18)\qquad K_\alpha(z) = \sqrt{\frac{\pi}{2z}}\exp(-z)\Big\{\sum_{k=0}^{\nu-1}(\alpha, k)(2z)^{-k} + \zeta_\nu(z)\Big\},\ \nu = 1, 2, 3, \dots,$$

in the region $\mathbb{C} \setminus (-\infty, 0]$, where the complex function ζ_ν is holomorphic in this region, and $\zeta_\nu(z) = O(z^{-\nu})$ provided ν is fixed and $z \to \infty$.

3. Hypergeometric functions

3.1 The equation

$$(3.1)\qquad z(1-z)w'' + [c - (a+b+1)z]w' - abw = 0$$

is called hypergeometric differential equation with parameters a, b, c. Its (analytic) solutions are called hypergeometric functions.

For $a \in \mathbb{C}$ we define

$$(3.2)\qquad (a)_0 = 1,\ (a)_k = a(a+1)(a+2)\dots(a+k-1),\ k = 1, 2, 3\dots,$$

i.e. if $a \ne 0, -1, -2, \dots$, then

$$(3.3)\qquad (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)},\ k = 0, 1, 2, \dots.$$

Suppose that $c \ne 0, -1, -2, \dots$, then the power series

$$(3.4)\qquad \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\cdot\frac{z^k}{k!}$$

converges in a neighbourhood of the origin, and its sum is a solution of the equation (3.1).

The analytic function which is defined by the power series (3.4) is called Gauss' hypergeometric function and usually it is denoted by $F(a,b;c;z)$. The same notation is used for the sum of the power series (3.4), i.e.

$$F(a,b;c;z) = 1 + \frac{ab}{c}\cdot\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\cdot\frac{z^2}{2!} + \dots. \tag{3.5}$$

Let us point out that if $a = -n$ (resp. $b = -n$), $n = 0,1,2,\dots,$ then the equation (3.1) has a solution which is a polynomial of degree n namely

$$F(-n,b;c;z) = \sum_{k=0}^{n} \frac{(-n)_k(b)_k}{(c)_k}\cdot\frac{z^k}{k!}. \tag{3.6}$$

Up to a constant factor this is the only polynomial solution of the equation (3.1). In all other cases every solution of this equation which is holomorphic in the neighbourhood of the zero point has a representation by an infinite power series centered at the origin. In particular, the series in the right-hand side of (3.5) (when $a, b \neq 0, -1, -2, \dots$) is called hypergeometric series. One can easily prove that the radius of convergence of each hypergeometric series is equal to one.

3.2 An equation of the kind

$$zw'' + (c-z)w' - aw = 0 \tag{3.7}$$

is called degenerate (confluent) hypergeometric differential equation and its (analytic) solutions are called degenerate (confluent) hypergeometric functions.

If $c \neq 0, -1, -2, \dots,$ then the power series

$$\sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\cdot\frac{z^k}{k!}$$

is convergent in the whole complex plane and, hence, it defines an entire function of the complex variable z. This function is a solution of the equation (3.5). Usually it is denoted by $\Phi(a;c;z)$, i.e.

$$\Phi(a;c;z) = 1 + \frac{a}{c}\cdot\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\cdot\frac{z^2}{2!} + \cdots. \tag{3.8}$$

The power series in the above equality is called the series of Kummer and (3.8) is known as Kummer's function with parameters a and c.

If $a = -n$, $n = 0,1,2,\dots,$ then the right-hand side of (3.8) reduces to a polynomial of n-th degree, i.e.

$$\Phi(-n;c;z) = \sum_{k=0}^{n} \frac{(-n)_k}{(c)_k}\cdot\frac{z^k}{k!}. \tag{3.9}$$

Moreover, up to a constant factor this is the only polynomial solution of the equation (3.7).

3.3 If $\operatorname{Re} a > 0$, then the integral

$$\Psi(a;c;z) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1}(1+t)^{c-a-1} \exp(-zt)\, dt \tag{3.10}$$

is absolutely uniformly convergent on every (closed) half-plane of the kind $\operatorname{Re} z \geq \delta > 0$. Hence, it defines a complex-valued function holomorphic in the right half-plane.

If $z = x > 0$, then replacing t by tx, we find that

$$\Psi(a;c;x) = \frac{x^{1-c}}{\Gamma(a)} \int_0^\infty \frac{t^{a-1}\exp(-t)}{(z+t)^{a-c+1}}\, dt.$$

The integral

$$I(a,c;z) = \int_0^\infty \frac{t^{a-1}\exp(-t)}{(z+t)^{a-c+1}}\, dt$$

is uniformly converegent on every compact subset of the region $\mathbb{C} \setminus (-\infty, 0]$, hence, it defines a holomorphic function there. Since $\Psi(a;c;x) = x^{1-c}(\Gamma(a))^{-1} I(a,c;x)$ for $x > 0$, it follows that the function (3.10) admits an analitical continuation in the region $\mathbb{C} \setminus (-\infty, 0]$ realized by the equality

$$\Psi(a;c;z) = \frac{z^{1-c}}{\Gamma(a)} \int_0^\infty \frac{t^{a-1}\exp(-t)}{(z+t)^{a-c+1}}\, dt, \ \ z \in \mathbb{C} \setminus (-\infty, 0]. \tag{3.11}$$

We call (3.11) the Tricomi degenerate (confluent) hypergeometric function with parameters a, c. It is easy to prove that it is a "second" solution of the equation (3.7) in the region $\mathbb{C} \setminus (-\infty, 0]$. More precisely, it means that the functions $\Psi(a;c;z)$ and $\Phi(a;c;z)$ are linearly independent in the region $\mathbb{C} \setminus (-\infty, 0]$.

3.4 If $\operatorname{Re} c > \operatorname{Re} a > 0$, then the integral representation

$$\Phi(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} \exp(zt)\, dt \tag{3.12}$$

holds. This can be easily verified using the power series expansion of the function $\exp(zt)$ as well as the defining equality (3.8).

Another integral representation of Kummer's function is the following:

$$\begin{aligned} & z^{c/2-1/2} \exp(-z)\Phi(a;c;z) \\ &= \frac{\Gamma(c)}{\Gamma(c-a)} \int_0^\infty t^{c/2-a-1/2} \exp(-t) J_{c-1}(2\sqrt{zt})\, dt. \end{aligned} \tag{3.13}$$

It holds in the region $\mathbb{C} \setminus (-\infty, 0]$ provided that $\operatorname{Re} c > \operatorname{Re} a > 0$.

Tricomi's function has an integral representation in terms of the modified Bessel functions of the third kind, i.e.

$$z^{c/2-1/2}\Psi(a;c;z) \tag{3.14}$$

$$= \frac{1}{\Gamma(a)\Gamma(a-c+1)}\int_0^\infty t^{a-c/2-1/2}\exp(-t)K_{c-1}(2\sqrt{zt})\,dt.$$

It holds in the region $\mathbb{C}\setminus(-\infty,0]$ provided that $\operatorname{Re} a>0$ and $\operatorname{Re}(a-c)>0$.

3.5 If $a\in\mathbb{C}\setminus(-\infty,0]$ and $c\neq 0,-1,-2,\dots$ is real, then there exist complex functions $\{\varphi_k(c;z)\}_{k=0}^\infty$ and $\{r_\nu(a,c;z)\}_{\nu=1}^\infty$ which are holomorphic in the region $\mathbb{C}\setminus(-\infty,0]$ and such that for $\nu=1,2,3\dots$ the representation

$$2\sqrt{\pi}(\Gamma(c))^{-1}(-z)^{c/2-1/4}\exp(-z/2)\Phi(a;c;z) \tag{3.15}$$

$$= a^{-c/2+1/4}\exp\{2(-z)^{1/2}\sqrt{a}\}\Big\{\sum_{k=0}^{\nu-1}\varphi_k(c;z)a^{-k/2}+r_\nu(a,c;z)\Big\}$$

holds in this region. Moreover, if ν and c are fixed, then for every positive number ρ, and every compact set $M\subset\mathbb{C}\setminus(-\infty,0]$ there exists a positive constant A $=A(\rho,M)$ such that $|a^{\nu/2}r_\nu(a,c;z)\le A$ when $\operatorname{Re} a^{1/2}\ge\rho$ and $z\in M$.

Suppose that a and c are real and that $a-c/2>0$. Then the representation

$$\sqrt{2}(a-c/2)^{a-c/2+1/4}z^{c/2-1/4}\exp(c/2-a-z/2)\Psi(a;c;z) \tag{3.16}$$

$$=\exp\{2(a-c/2)^{1/2}\sqrt{z}\}\{1+\psi(a,c;z)\}$$

holds in the region $\mathbb{C}\setminus(-\infty,0]$, where $\psi(a,c;z)$ is a holomorphic function in this region. Moreover, if c is fixed, then for every $\rho>\max(0,c/2)$ and every compact set $E\subset\mathbb{C}\setminus(-\infty,0]$ there exists a constant $B=B(\rho,E)>0$ such that $|a^{-1/2}\psi(a,c;z)|\le B$ for $a\ge\rho$ and $z\in E$.

There is another asymptotic formula for the function $\Psi(a;c;z)$ when a,c are fixed and $z\to\infty$ in the region $\mathbb{C}\setminus(-\infty,0]$. More precisely, for every $\nu=0,1,2,\dots$ the representation

$$\Psi(a;c;z)=\sum_{k=0}^{\nu}(-1)^k\frac{(a)_k(a-c+1)_k}{k!}z^(-a-k)+\theta_\nu(a,c;z) \tag{3.17}$$

holds where the complex functions $\{\theta_\nu(a,c;z)\}_{\nu=0}^\infty$, $\nu=0,1,2,\dots$, which are holomorphic in the region $\mathbb{C}\setminus(-\infty,0]$, are such that $|z^{-a-\nu-1}\theta_\nu(a,c;z)|=O(1)$ when $z\to\infty$ in this region.

4. Weber-Hermite functions

4.1 The equation

$$w'' + (\nu + 1/2 - z^2/4)w = 0, \ \nu \in \mathbb{C} \tag{4.1}$$

is called the Weber-Hermite differential equation. Its solutions are called functions of the parabolic cylinder. All they are holomorphic in the whole complex plane, i.e. all they are entire functions of the complex variable z. Moreover, they can be expressed in terms of the degenerate hypergeometric functions. In particular, the function

$$\begin{aligned} D_\nu(z) &= 2^{(\nu-1)/2} \exp(-z^2/4) z \Psi((1-\nu)/2; 3/2; z^2/2) \\ &= 2^{\nu/2} \exp(-z^2/4) \left\{ \frac{\Gamma(1/2)}{\Gamma((1-\nu)/2)} \Phi(-\nu/2; 1/2; z^2/2) \right. \\ &\left. + (z/\sqrt{2}) \frac{\Gamma(-1/2)}{\Gamma(-\nu/2)} \Phi((1-\nu)/2; 3/2; z^2/2) \right\} \end{aligned} \tag{4.2}$$

is a solution of the equation (4.1). It is called the Weber-Hermite function with index (parameter) ν.

Other solutions of the equation (4.1) are the function $D_\nu(-z), D_{-\nu-1}(iz)$, and $D_{-\nu-1}(-iz)$. The last two of them are linearly independent, hence, all the solutions of (4.1) are their linear combinations. In particular,

$$D_\nu(z) = \frac{\Gamma(\nu+1)}{\sqrt{2\pi}} \{\exp(\nu\pi i/2) D_{-\nu-1}(is) + \exp(-\nu\pi i/2) D_{-\nu-1}(-iz)\}. \tag{4.3}$$

As a corollary of (4.2), one can prove that if $\nu = n$ is a nonnegative integer, then the function

$$\exp(z^2/4) D_n(z) \tag{4.4}$$

is a polynomial of n-th degree. Moreover, up to a constant factor, (4.4) is the only polynomial solution of the equation (4.1) with $\nu = n, \ n = 0, 1, 2, \ldots$.

4.2 There are many integral representations for the Weber-Hermite functions. Further we include only those used in this book. Each of them holds in the whole complex plane provided the parameter ν satisfies an additional restriction.

Suppose that $\operatorname{Re} \nu > -1$, then

$$D_\nu(z) = \sqrt{\frac{2}{\pi}} \exp(z^2/4) \int_0^\infty t^\nu \exp(-t^2/4) \cos(zt - \nu\pi/2)\, dt. \tag{4.5}$$

If $\operatorname{Re} \nu < 0$, then

$$D_\nu(z) = \frac{\exp(-z^2/4)}{\Gamma(-\nu)} \int_0^\infty t^{-\nu-1} \exp(-t^2/2 - zt)\, dt. \tag{4.6}$$

If c is an arbitrary positive number and ν is an arbitrary complex number, then

$$D_\nu(z) = \frac{1}{\sqrt{2\pi}} \exp(z^2/4 - \nu\pi i/2) \int_{ic-\infty}^{ic+\infty} \exp(-\zeta^2/2 + iz\zeta)\zeta^\nu \, d\zeta. \tag{4.7}$$

4.3 For each $z \in \mathbb{C}$ and $\nu \in \mathbb{C} \setminus [0, \infty)$ we have the representation

$$D_\nu(z) = \frac{1}{\sqrt{2}}(-\nu)^{\nu/2} \exp(-\nu/2) \exp\{-(-\nu)^{1/2} z\}\{1 + \delta(z, \nu)\}, \tag{4.7}$$

where $\delta(z, \nu)$ is a holomorphic function of the complex variables z and ν in the region $\mathbb{C} \times \{\mathbb{C} \setminus [0, \infty)\}$. Moreover, if $0 < \rho < R < \infty$, then there exists a positive constant $M = M(\rho, R)$ such that $|(-\nu)^{1/2}\delta(z, \nu)| \le M$ for every $z \in U(0; R)$ and every ν such that $\mathrm{Re}(-\nu) \ge 0$, and $|\nu| \ge \rho$. In particular, it follows that $\lim_{|\nu|\to\infty} \delta(z, \nu) = 0$ uniformly on every compact subset of the complex plane provided that $\mathrm{Re}(-\nu) \ge 0$.

Comments and referneces

The fundamential work of H. Bateman and A. Erdélyi [1] is the main source of information about classical special functions used in this book.

Of course, there are some "slight" modifications. E.g., we prefer the explicit analytical continuation (3.11) of the Tricomi function in the region $\mathbb{C} \setminus (-\infty, 0]$ instead of that given as [H. Bateman, A. Erdélyi, 1, 6.5, (3)].

The asymptotic expansion (3.15) for the Kummer function in the region $\mathbb{C}$ $\setminus[0, \infty)$ is due to O. Perron [1]. Here it is in the form given by [E. Hille, 4, (1.15)].

The asymptotic formula (4.7) is due to T.M. Cherry [1]. A more general asymptotic expansion for the Weber-Hermite function $D_\nu(z)$ involving a complex parameter is given in [P. Rusev, 33]. The last paper contains a generalization of Cherry's formula as well as of the asymptotic formulas of Szegö's type for the Hermite polynomials.

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Author index

Subject index